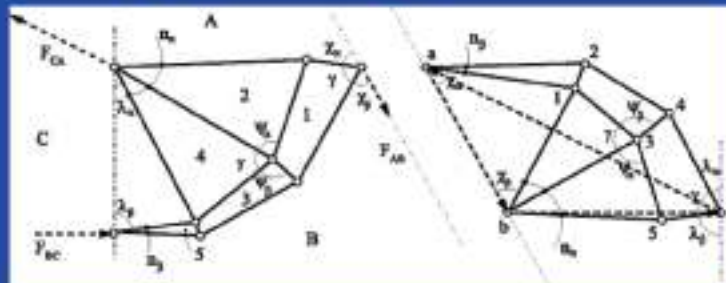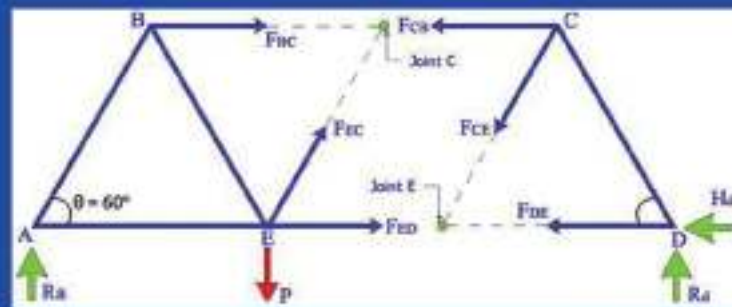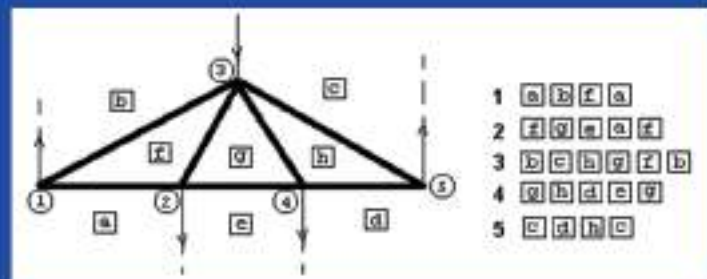# G.C.E. Advanced Level

Combined Mathematics
STATICS - II
Additional Reading Book

(Prepared According to the New syllabus Implemented From 2017)

Department of Mathematics
National Institute of Education
Maharagama
Sri Lanka
www.nie.lk

G.C.E. Advanced Level

Combined Mathematics

STATICS - II

Additional Reading Book

Combined Mathematics
Statics - II
Additional Reading Book

Department of Mathematics
Faculty of Science and Technology
National Institute of Education
Maharagama

Message from the Director General

Department of Mathematics of National Institute of Education time to time implements many different activities to develop the mathematics education. The publication of this book is a mile stone which was written in the name of **"Statics - Part I, Statics - Part II".**

After learning of grade 12 and 13 syllabus, teachers should have prepared the students for the General Certificate of Education (Advanced Level) which is the main purpose of them. It has not enough appropriate teaching - learning tools for the proper utilization. It is well known to all, most of the instruments available in the market are not appropriate for the use and it has not enough quality in the questions. Therefore **"Statics - Part I, Statics - Part II".** book was prepared by the Department of Mathematics of National Institute of Education which was to change of the situation and to ameliorate the students for the examination. According to the syllabus the book is prepared for the reference and valuable book for reading. Worked examples are included which will be helpful to the teachers and the students.

I kindly request the teachers and the students to utilize this book for the mathematics subjects' to enhance the teaching and learning process effectively. My gratitude goes to Aus Aid project for sponsoring and immense contribution of the internal and external resource persons from the Department of Mathemetics for toil hard for the book of **"Statics - Part I, Statics - Part II".**

Dr. (Mrs). T. A. R. J. Gunasekara
Director General
National Institute of Education.

Message from the Director

Mathematics holds a special place among the G.C.E. (A/L) public examination prefer to the mathematical subject area. The footprints of the past history record that the country's as well as the world's inventor's spring from the mathematical stream.

The aim and objectives of designing the syllabus for the mathematics stream is to prepare the students to become experts in the Mathematical, Scientific and Technological world.

From 2017 the Combined Mathematics syllabus has been revised and implemented. To make the teaching - learning of these subjects easy, the Department of Mathemactics of National Institute of Education has prepared **Statics - Part 1 and Part 11** as the supplementary reading books. There is no doubt that the exercises in these books will measure their achievement level and will help the students to prepare themselves for the examination. By practicing the questions in these books the students will get the experience of the methods of answering the questions. Through the practice of these questions, the students will develop their talent, ability, skills and knowledge. The teachers who are experts in the subject matter and the scholars who design the syllabus, pooled their resources to prepare these supplementary reading books. While preparing these books, much care has been taken that the students will be guided to focus their attention from different angles and develop their knowledge. Besides, the books will help the students for self-learning.

I sincerely thank the Director General for the guidance and support extented and the resource personnel for the immense contribution. I will deeply appreciate any feedback that will shape the reprint of the books.

Mr. K. R. Pathmasiri
Director
Department of Mathematics

Curriculum Committee

Approval: **Academic Affairs Board**
National Institute of Education

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National Institute of Education

Preface

This book is being prepared for the students of Combined Mathematics G.C.E.A/L to get familiar with the subject area of Statics. It is a supplementary book meant for the students to get practice in answering the questions for self learning. The teachers and the students are kindly invited to understand, it is not a bunch of model questions but a supplementary to encourage the students towards self learning and to help the students who have missed any area in the subject matter to rectify them.

The students are called upon to pay attention that after answering the questions in worked examples by themselves, they can compare their answers with the answers given in the book. But it is not necessary that all the steps have taken to arrive at the answers should tally with the steps mentioned in the book's answers given in this book are only a guide.

Statics Part 11 is released in support of the revised syllabus - 2017. The book targets the students who will sit for the GCE A/L examination – 2019 onwards. The Department of Mathematics of National Institute of Education already released **Practice Questions and Answers** book and book of **'Statics – I ',** it is being proceeded by the **" Statics II".** There are other two books soon be released with the questions taken Unit wise **"Questions bank".**

We shall deeply appreciate your feedback that will contribute to the reprint of this book.

Mr.S.Rajendram
Project Leader
Grade 12, 13 Maths
National Institute of Education.

Content

5.0 Jointed Rods

In previous chapters 4.1 and 4.2 we considered the action of coplanar forces acting on a single rigid body In this chapter we shall consider the action of coplanar forces acting on two or more rigid bodies, specially on number of heavy rods jointed together to form a frame.

We will consider the equilibirium of rods under the action of their weight, any external forces applied and forces exerted on their ends by hings (joints)

5.1 Types of simple joints

(i)   Rigid Joint

When two rods are jointed together such a way that they cannot be seperated or turn about one another at the joint, the joint is rigid joint.

(ii)   Pin Joint

When two rods are jointed by a light pin such a way that they can turn at the joint, the joint is pin joint if they can turn freely at the joint, the joint is a smooth pin joint and, if free turn is not possible the joint is a rough joint.

We shall consider the frames with smooth pin joints in this chapter.
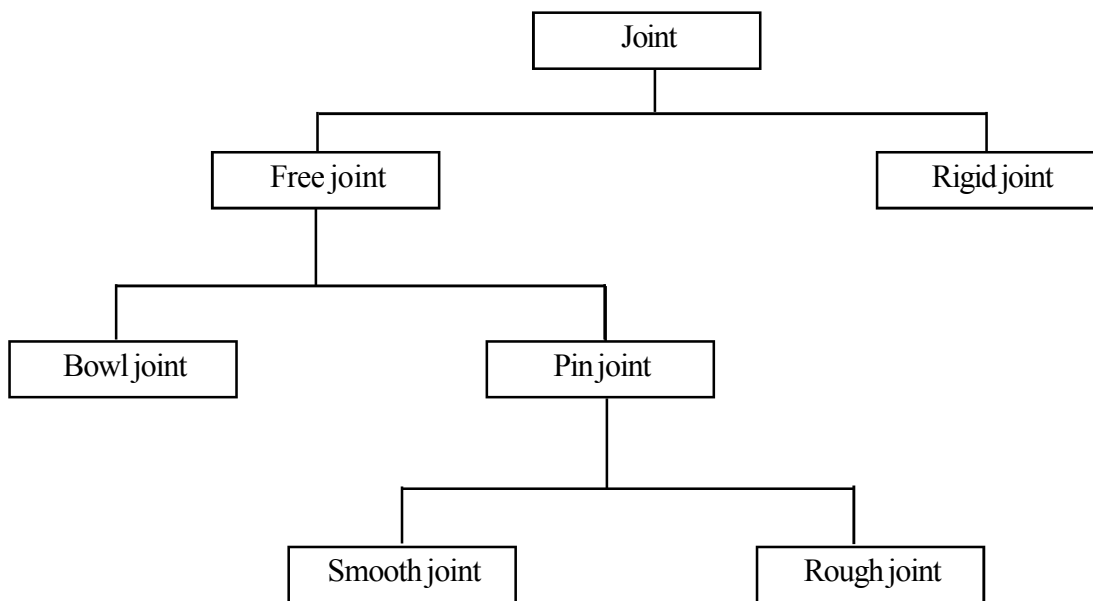


5.2 Rigid joint

If the shape of a body obtained by joining two or more bodies together cannot be changed by external forces then the joint is said to be a rigid joint.

Force at a smooth joint (Pin joint)

The joints are shown seperately to show the reaction at the joint. The reaction at the joints will be equal and opposite. To find R easily the components of R are shown as follows.
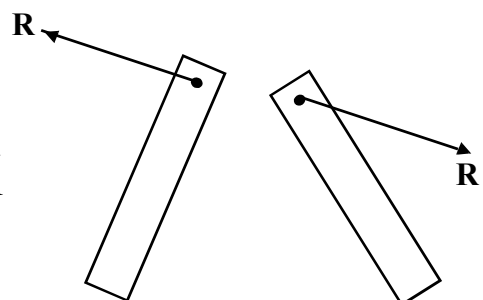
X, Y are the horizontal and vertical components of R. R is the resultant of X and Y, and passes through the pin joint.
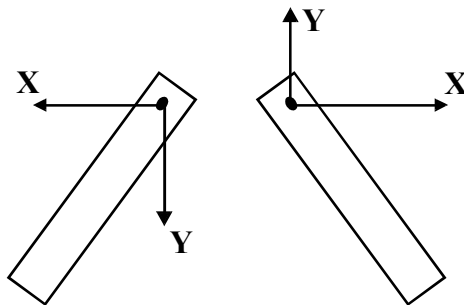

The light pin is assumed to be a small smooth pin of circular rim, joins the rod by passing through the rods. As the pin is smooth the reaction on contact is perpendicular to it and for rods. Since the pin is in equilibirium under these two forces, they are equal opposite in direction and have the sameline of action. Therefore the reaction on each rod is equal and opposite and have the same line of action an each joints.

For conveneience we resolve the reaction into two perpendicular components when we need of it.

Note :

When a heavy rod is joined at its ends to another rod, the reaction by joints on the rod cannot lie along the rod, since the rod is acted by three forces.
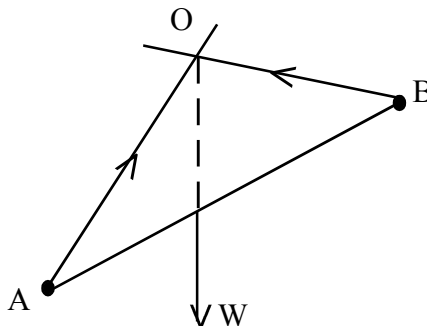
For equilibrium forces should meet at one point O which cannot lie on AB.

If the rod is light, it is acted by the two reactions only, so that they always lie along the rod to balance each other.

When a framework is symmetric about an axis identical set of forces will act on both sides.

Instructions to solve problems

(i)   Correct diagram has to be drawn with geometrical data.

(ii) Forces should be marked correctly.

(iii) Necessary equations should be obtained to find the unknown forces.

(iv) To find the reactions at a joint the force at the joint should be divided into two components and to be marked.

(If there is axis of symmetry, it should be stated and the results can be used)

Note :

A framework must be rigid. To make a framework of n jonts (n > 3) to be rigid it is necessary to have (2n-3) rods.

A framework with more than (2n-3) rods will make the framework over rigid.

5.3 Worked examples

Example 1

Three uniform equal rods of length $2a$ and weight W are freely jointed at their end points and the frame ABC is suspended from the joint A. Find the magnitude and direction of the reaction at B on AB.
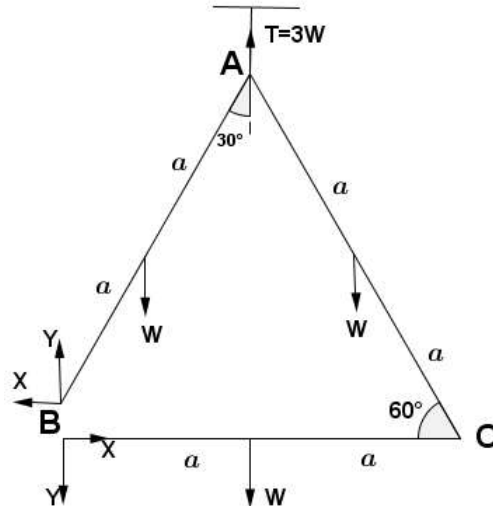
Consider the equilibrium of BC

Taking moments about C for BC

$$\text{cm} \qquad W.a + Y.2a = 0$$

$$2Y + W = 0 \qquad ; Y = -\frac{W}{2}$$

Consider the equilibrium of AB.

Taking momements about A for AB

$$\text{Am} \qquad Y(2a \sin 30°) + X (2a \cos 30°) - W(a \sin 30°) = 0$$

$$2Y + 2X \cot 30° = W$$

$$2Y + 2\sqrt{3}X = W$$

$$-W + 2\sqrt{3}X = W \qquad ; X = \frac{W}{\sqrt{3}}$$

$$R = \sqrt{X^2 + Y^2} = \sqrt{\frac{W^2}{3} + \frac{W^2}{4}} = \sqrt{\frac{7}{12}}W$$

$$\tan \theta = \frac{Y}{X} = \frac{\sqrt{3}}{2} ; \qquad \therefore \ \theta = \tan^{-1}\left(\frac{\sqrt{3}}{2}\right)$$

Magnitude of the reaction at $B = \sqrt{\frac{7}{12}}\ W$; R makes an angle θ with CB where $\theta = \tan^{-1}\left(\frac{\sqrt{3}}{2}\right)$

Example 2

Two uniform rods AB, AC each of length $2a$ and weight W are smoothly jointed at A. The rods are in equilibrium in a vertical plane with B and C lying on a smooth horizontal plane and C is connected to the midpoint of AB by an inextensible string and $\hat{BAC} = 60°$. Find the tension in the string and the reaction at A.
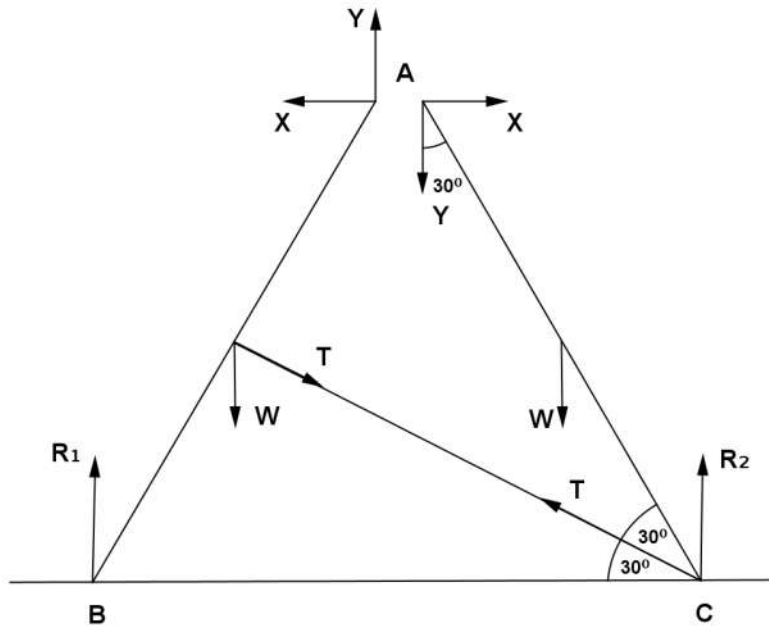
$AB = AC$; $\hat{BAC} = 60°$

Therefore ABC is an equilateral triangle.

For equilibrium of AB and AC,

Resolving vertically

$\uparrow$ $R_1 + R_2 - 2W = 0$; $R_1 + R_2 = 2W$ .... ①

Taking moment about C

Cm $\quad$ -R_1 . $4a \cos 60° + W$. $a \cos 60° + W$. $3a \cos 60° = 0$ .... ②

$\quad R_1 = W$ and $\quad R_2 = W$

For equilibrium of AC,

Am $\quad$ -$W . a \cos 60° -$ T . $a + R_2$. $2a \cos 60° = 0$ .... ③

$\quad -\dfrac{W}{2} - T + W = 0$; $\quad T = \dfrac{W}{2}$

For equilibrium of AC, moment about A

Resolving horizontally,

$\rightarrow X - T \cos 30° = 0$; $X = T \cos 30° = \dfrac{\sqrt{3}W}{4}$

Resolving vertically,

$\uparrow R_2 - Y - W + T \sin 30° = 0$

$Y = R_2 - W + \dfrac{T}{2} = \dfrac{W}{4}$

Hence reaction at A is $\sqrt{X^2 + Y^2} = \sqrt{\dfrac{3W^2}{16} + \dfrac{W^2}{16}} = \dfrac{W}{2}$

Example 3

Two uniform equal rods AB, AC each of weight W are smoothly jointed at A. The ends B and C rest on a horizontal smooth plane and the frame ABC is kept in a vertical plane. The equilibrium is maintained by connecting midpoints of AB and AC by an inextensible string. If $B\hat{A}C = 2\theta$, find the tension in the string and the magnitude of the reaction at A on AB.

Let $AB = AC = 2a$

For the equilibrium of AB and AC,

Resolving vertically,

$\uparrow 2R - 2W = 0$

$R = W$ .... ①

For equilibrium of AB,

Resolving vertically,

$\uparrow R + Y - W = 0$

$W + Y - W = 0 ; Y = 0$ .... ②

Resolving horizontally,

$\rightarrow T - X = 0 ; T = X$ .... ③

Taking moment about A for equilibrium of AB,

A↻ $T.a\cos\theta + W.a\sin\theta - R.2a\sin\theta = 0$

$$T = \frac{(2W - W)\sin\theta}{\cos\theta} = W\tan\theta \quad .... ④$$

Reaction at A is $W\tan\theta$
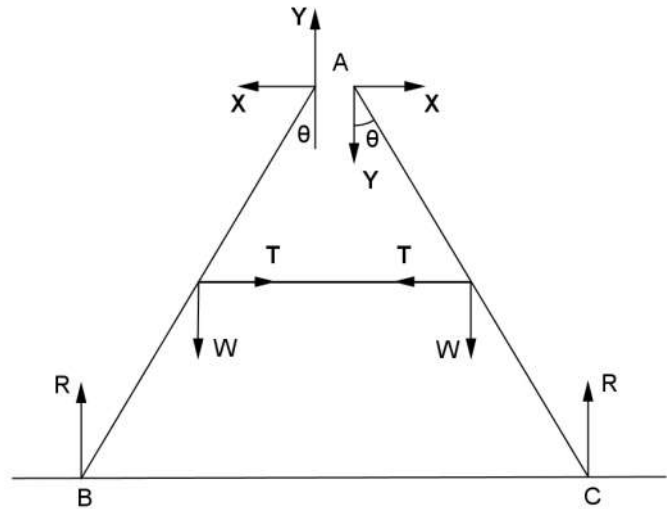
Note :

In the above example the system is symmetrical about the vertical axis through A

Now the reaction at A is given by

Since the system is symmetri about the vertical axis through A, the forces should be as given below.

Hence $Y = 0$

Example 4

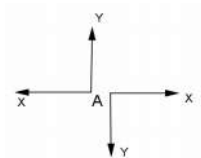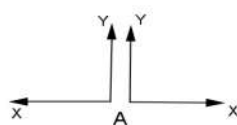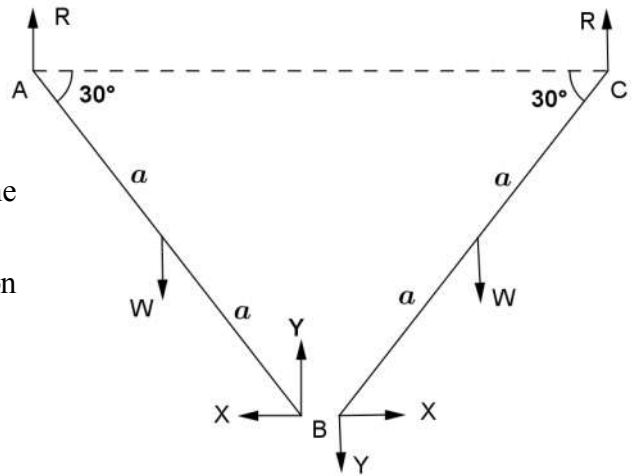AB, BC are two uniform rods each of lengh $2a$ and weight W, smoothly hinged at B, and the frame ABC is suspended from the points A and C at the same horizontal level. The systerm is in a vertical plane and each rod makes 30° with the horizontal. Find the reaction at the joint B.

The system is symmetrical about the vertical line through B.

Therefore the vertical component (Y) of the reaction at B is zero (Y=0)

For the equilibrium of AB

By taking moments about A

$$Am \quad -X . 2a \sin 30° + Y . 2a \cos 30° - Wa \cos 30° = 0$$

$$-X . 2a \sin 30° = W.a \cos 30°$$

$$X = -\frac{\sqrt{3}W}{2}$$

Example 5

AB, BC are two equal uniform rods each of length $2a$ and weight W and $2W$ respectively. The rods are smoothly jointed at B and the frame ABC is suspended from A and C at the same horizontal level. The system is in the vertical plane and each rod makes 60° with the horizontal. Find the magnitude and the direction of the reaction at the joint B on AB.

For equilibrium of the system

Resolving horizontally,

$$\rightarrow X_1 - X_2 = 0 ; \quad X_1 = X_2$$

Resolving vertically

$$\uparrow R_1 + R_2 - 3W = 0 \quad ; \quad R_1 + R_2 = 3$$
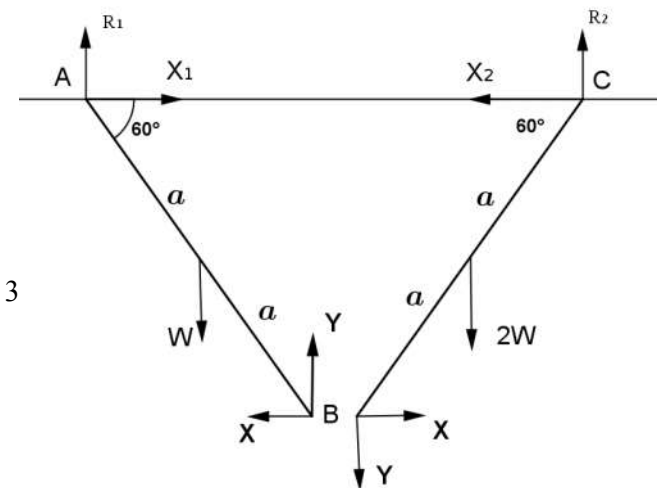
For AB and AC moment about A

$$Am \quad R_2 . 2a - W . \frac{a}{2} - 2W . \frac{3a}{2} = 0$$

$$2R_2 = \frac{7W}{2} ; \quad R_2 = \frac{7W}{4} \text{ and } R_1 = \frac{5W}{4}$$

For equilibrium of BC,

Resolving vertically $\uparrow$ $R_2 - 2W - Y = 0$; $Y = R_2 - 2W = \frac{7W}{4} - 2W = -\frac{W}{4}$

For equilibrium of BC

Cm $X . 2a \sin 60° + Y . 2a \cos 60° + 2W . a \cos 60° = 0$

$X. 2a \sin 60° - \dfrac{W}{4}. 2a \cos 60° + 2W. a \cos 60° = 0$
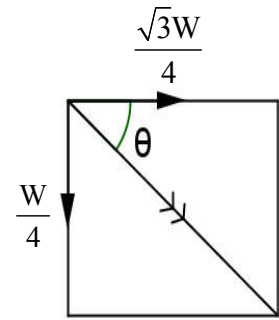
$$X = - \frac{\sqrt{3}W}{4}$$

$$R = \sqrt{\frac{3W^2}{16} + \frac{W^2}{16}}$$

$$R = \frac{W}{2}$$

$$\tan \theta = \frac{\dfrac{W}{4}}{\dfrac{\sqrt{3}W}{4}} = \frac{1}{\sqrt{3}}$$

$$\theta = \tan^{-1}\left(\frac{1}{\sqrt{3}}\right)$$

$$= \frac{\pi}{6}$$

Example 6

Three uniform equal rods AB, BC, AC each of length $2a$ and weight W are smoothly jointed at their ends to form an equilateral triangle. The frame is freely hinged at A in a vertical plane. The triangle is kept in equilibrium with AB as horizontal and C is below AB by a force at B perpendicular to BC by a force P at B perpendicular to BC. AB is horizontal and C is below AB. Find the value of P. Also find the horizontal and vertical components of the horizontal and vertical components of the reaction at C.

By taking moments about A for the system

Am $- W .a \cos 60° - W .a - W (2a - a \cos 60°) + P. 2a \cos 60° = 0$

$P = 3W$

By taking moment about A for equilibrium of AC,

Am $X. 2a \sin 60° + Y. 2a \cos 60° + W. a \cos 60° = 0$

$\Rightarrow X + \dfrac{Y}{\sqrt{3}} = -\dfrac{W}{2\sqrt{3}}$ .... ①

By taking moments about B for equilibrium of BC,

Bm $X . 2a \sin 60° - Y . 2a \cos 60° + W . a \cos 60° = 0$

$\Rightarrow X - \dfrac{Y}{\sqrt{3}} = -\dfrac{W}{2\sqrt{3}}$ .... ②

7
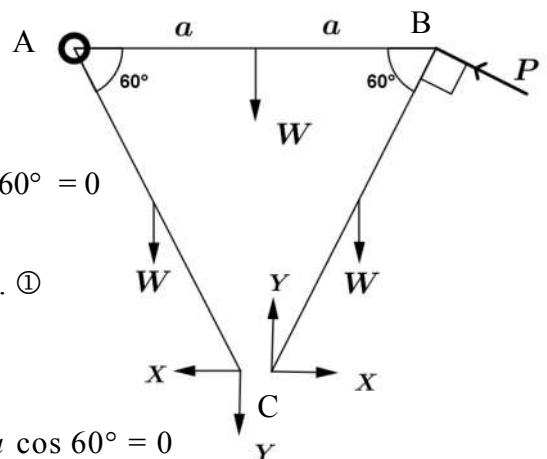
From ① and ②

$$Y = 0$$

$$X = -\frac{W}{2\sqrt{3}}$$

Reaction at C is $\dfrac{W}{2\sqrt{3}}$

Example 7

Two uniform rods AB and BC each of length $2a$ and weights 2W, W respectively are smoothly hinged at B. The mid points of the rods are connected by a light in inelastic string. The system in a vertical plane with other ends A and C lie on a smooth horizontal table. If $A\hat{B}C = 2\theta$ show that the tension in the string is

$\dfrac{3\,W}{2}\tan\theta$. Find the magnitude and direction of the reaction at B.

For the equilibrium of the system,

By taking moments about C

$$\mathcal{C}\mathrm{m} \qquad W.\,a\sin\theta + 2W.\,3a\sin\theta - R.\,4a\sin\theta = 0$$

$$R = \frac{7W}{4}$$

For equilibrium of AB , taking moment about B

$$\mathcal{B}\mathrm{m} \qquad T.\,a\cos\theta + 2W.\,a\sin\theta - R.\,2a\sin\theta = 0$$

$$T = -2W\tan\theta + 2R.\tan\theta$$

$$T = -2W\tan\theta + \frac{7W}{2}\tan\theta$$

$$T = \frac{3W}{2}\tan\theta$$

For equilibrium of AB

Resolving horizontally,

$$\rightarrow T - X = 0 \ ; \ X = T = \frac{3W}{2}\tan\theta$$
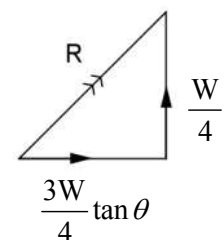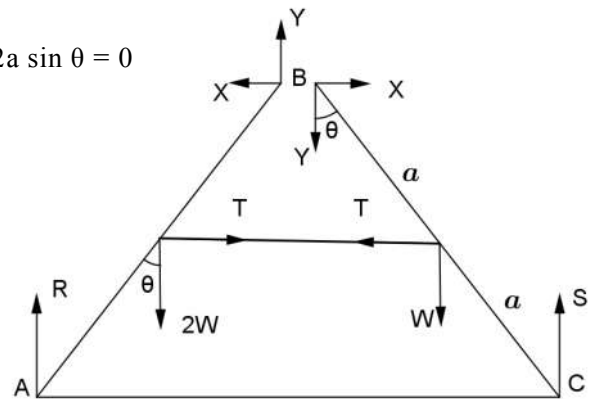
Resolving vertically,

$$\uparrow \ Y + R - 2W = 0$$

$$Y = 2W - \frac{7W}{4} = \frac{W}{4}$$

$$R = \sqrt{X^2 + Y^2}$$

$$= \sqrt{\frac{9W^2}{4}\tan^2\theta + \frac{W^2}{16}}$$

$$= \frac{W}{4}\sqrt{1 + 36\tan^2\theta}$$

Example 8

AB, BC, CD, DE are four uniform equal rods of length $2a$, smoothly jointed at B, C and D. The weights of AB, DE are $2W$ each and the weights of BC, CD are W each. The system is suspended from A and E at the same horizontal level. AB and BC make α, β with the vertical respectively. Show that the reaction at C is horizontal and the magnitude is $\dfrac{W}{2}\tan\beta$. Show also that $\tan\beta = 4\tan\alpha$.
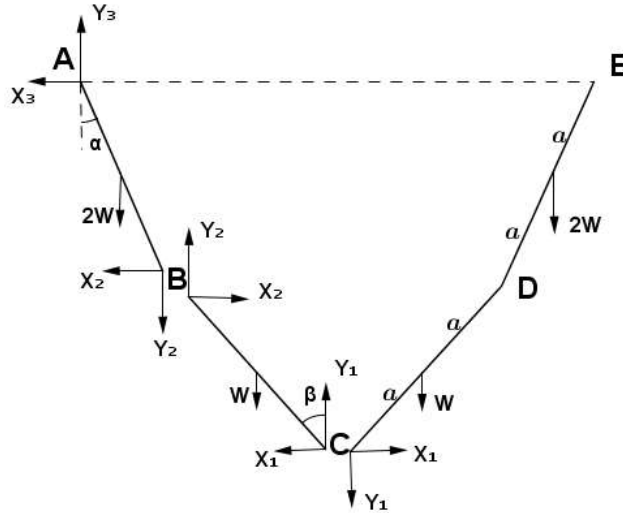
The system is symmetrical about the vertical axis through C

Therefore the vertical component of the reaction at C is zero. $Y_1 = 0$

For the equilibrium of BC

Resolving the forces horizontally

$$\rightarrow X_1 - X_2 = 0$$

$$X_1 = X_2$$

Resolving the forces vertically

$$\uparrow Y_1 + Y_2 - W = 0$$

$$Y_2 = W$$

moment about B

$$Bm \quad -X_1 \cdot 2a\cos\beta - W \cdot a\sin\beta = 0$$

$$X_1 = -\frac{W}{2}\tan\beta$$

For equilibrium of AB,

$$Am \quad -X_2 \cdot 2a\cos\alpha + 2W \cdot a\sin\alpha + Y_2 \cdot 2a\sin\alpha = 0$$

$$X_2 = -2W\tan\alpha$$

$$X_1 = X_2$$

$$\frac{W}{2}\tan\beta = 2W\tan\alpha$$

$$\tan\beta = 4\tan\alpha$$

Example 9

Two equal uniform rods AB and AC each of weight W are freely jointed at A, and the ends B and C are connected by a light inextensible string. The rods are kept in equilibrium in a vertical plane with the ends B and C on two smooth planes each of which inclined at an angle α to the horizontal; BC being horizontal and A being above BC. Find the reaction at B. If $\tan \theta > \tan 2\alpha$, where $\hat{BAC} = 2\theta$ then show that the tension in the string is $\frac{1}{2}W(\tan \theta - 2 \tan \alpha)$. Find also the reaction at the joint A.
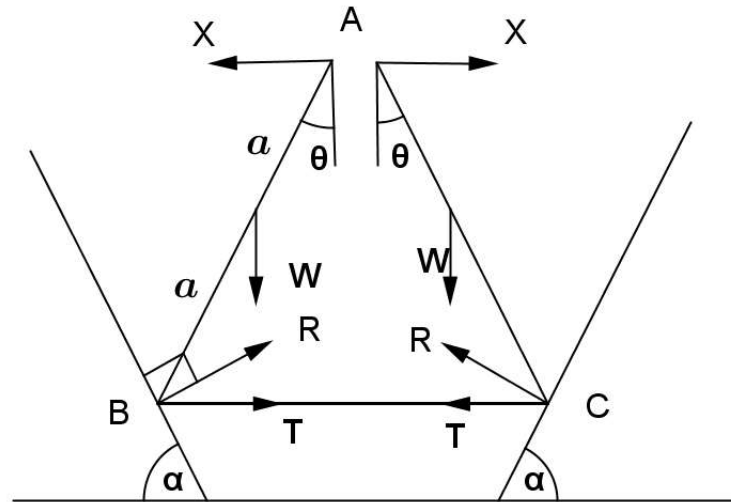
Let $2a$ be the length of each rod.

The system is symmetrical about the vertical axis through A.

Hence the vertical component of the reaction at A is zero.

For equilibrium of the system

Resolving vertically,

$\uparrow \ 2R \cos \alpha - 2W = 0 \ ; \ R = W \cos \alpha$

For equilibrium of AB, Taking moment about A

$A\text{m} \quad T. 2a \cos \theta + R \sin \alpha . 2a \cos \theta + W . a \sin \theta - R \cos \alpha . 2a \sin \theta = 0$

$$T = \frac{W}{2}(\tan \theta - 2 \tan \alpha)$$

For equilibrium of AB,

$B\text{m} \quad X . 2a \cos \theta - W . a \sin \theta = 0$

$$X = \frac{W}{2} \tan \theta$$

Example 10

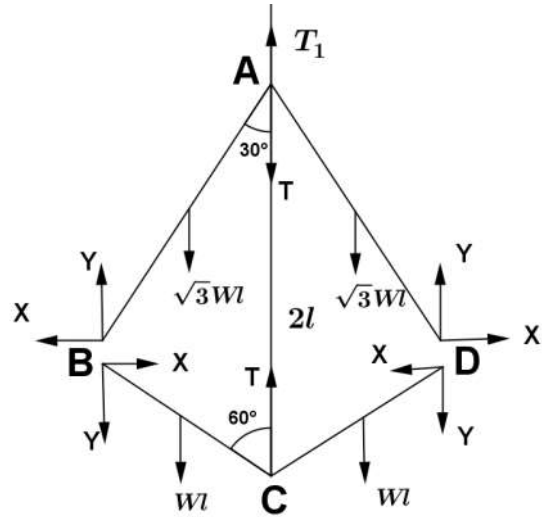AB, BC, CD and AD are four uniform rods having lengths $AB = AD = \sqrt{3}\ell$ and $BC = DC = \ell$ and are smoothly jointed at their ends to form a frame ABCD. The rods have weights W per unit length. The joints A and C are connected by an inelastic string of length 2ℓ. The frame is suspended in a vertical plane from A. Show that the tension in the string is $\dfrac{W\ell}{4}\left(\sqrt{3}+5\right)$

Method 1

$AB^2 + BC^2 = 3\ell^2 + \ell^2 = 4\ell^2 = AC^2$

Therefore, $A\hat{B}C = 90°$, $B\hat{A}C = 30°$, $B\hat{C}A = 60°$

The system is symmetrical about AC. Hence reactions at B and D are same.

For equilibrium of AB, Taking moment about A

$$A\text{m} \quad X.\sqrt{3}\ell\cos 30° + Y.\sqrt{3}\ell\sin 30° - \sqrt{3}\ell W.\frac{\sqrt{3}}{2}\ell\sin 30° = 0$$

$$X.\cot 30° + Y = \frac{\sqrt{3}}{2}\ell W$$

$$\sqrt{3}X + Y = \frac{\sqrt{3}}{2}\ell W \quad \text{................................} \quad ①$$

For equilibrium of BC, moment about C

$$C\text{m} \quad Y.\ell\sin 60° + W\ell.\frac{\ell}{2}\sin 60° - X.\ell\cos 60° = 0$$

$$Y + \frac{W\ell}{2} = \frac{X}{\sqrt{3}}$$

$$X = \sqrt{3}Y + \frac{\sqrt{3}W\ell}{2} \quad \text{....................} \quad ②$$

Substitute ① and ②

$$Y + \sqrt{3}X = \frac{\sqrt{3}W\ell}{2}$$

$$Y + \sqrt{3}\left(\sqrt{3}Y + \frac{\sqrt{3}W\ell}{2}\right) = \frac{\sqrt{3}W\ell}{2}$$

$$4Y + \frac{3W\ell}{2} = \frac{\sqrt{3}W\ell}{2}$$

$$Y = \frac{W\ell}{8}\left(\sqrt{3}-3\right)$$

for equilibrium of BC and CD

$$\uparrow \ T - 2Y - 2W\ell = 0$$

$$T = 2Y + 2W\ell$$

$$T = 2\frac{W\ell}{8}\left(\sqrt{3}-3\right) + 2W\ell$$

$$T = \frac{W\ell}{4}\left(\sqrt{3}+5\right)$$

or For BC and CD take moments about D

Method 2

$$AB^2 + BC^2 = 3\ell^2 + \ell^2 = 4\ell^2 = AC^2$$

$$A\hat{B}C = 90^0 \ , \ B\hat{A}C = 30^0 \ , \ B\hat{C}A = 60^0$$

By symmetry reactions at B and D are same.

The components of the reaction at B are taken along BA and BC, since $A\hat{B}C = 90^0$

For the equilibrium of the rod AB,

$$A\text{m} \quad \sqrt{3}W\ell . \frac{\sqrt{3}\ell}{2}\sin 30° - Y.\sqrt{3}\ell = 0$$

$$Y = \frac{\sqrt{3}W\ell}{4}$$

For the equilibrium of BC,

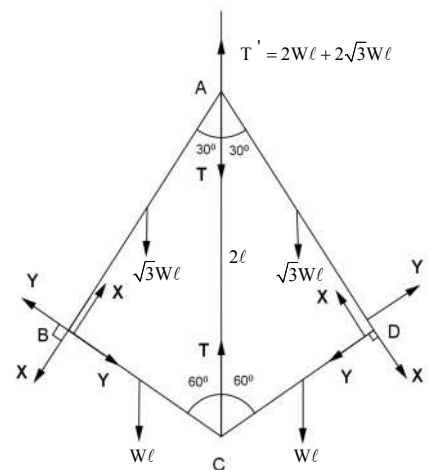$$C\text{m} \quad W\ell . \frac{\ell}{2}\sin 60 - X.\ell = 0$$

$$X = \frac{\sqrt{3}W\ell}{4}$$

For the equilibrium of BC and CD, Resolving vertically

$$T - 2W\ell + 2X \cos 30° - 2Y \cos 60° = 0$$

$$T = 2W\ell + 2Y \cos 60° - 2X \cos 30°$$

$$= 2W\ell + \frac{\sqrt{3}W\ell}{4} - \sqrt{3}.\frac{\sqrt{3}W\ell}{4}$$

$$= \frac{W\ell}{4}\left(\sqrt{3}+5\right)$$

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Example 11

Four uniform rods AB, BC, CD, DA each of length $2a$ and weight W are freely hinged at their ends, and rest with the upper rods AB, AD in contact with two smooth pegs in the same horizontal line at a distance $2c$ apart. If the inclination of the rods to the vertical is θ, determaine the horizontal and vertical components of the reaction at B and prove that $c = 2a \sin^3 \theta$.
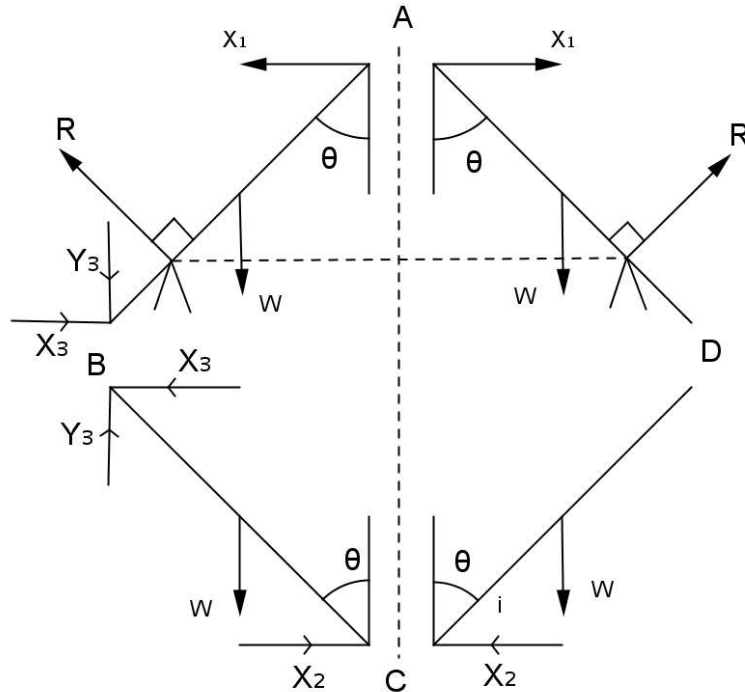
The system is symmetrical about AC. Therefore, the vertical components of the reaction at A and C are zero.

For equilibrium of the system,

Resolving vertically,

$$\uparrow 2R \sin \theta - 4W = 0$$

$$R = \frac{2W}{\sin \theta} \quad\text{.......................} \ ①$$

For equilibrium of BC,

$$\text{B}m \quad X_2. \ 2a \cos \theta - W.a \sin \theta = 0$$

$$X_2 = \frac{W \tan \theta}{2} \quad\text{...............} \ ②$$

Resolving horizontally,

$$\rightarrow X_2 - X_3 = 0; \quad X_3 = X_2 = \frac{W \tan \theta}{2} \quad\text{......} \ ③$$

Resolving vertically

$$\uparrow Y_3 - W = 0 \ ; \quad Y_3 = W \quad\text{.................} \ ④$$

For equilibrium of AB,

$$\text{A}m \quad -R.\frac{c}{\sin \theta} + Wa. \sin \theta + Y_3.2a \sin \theta + X_3.2a \cos \theta = 0$$

$$-\frac{2W.c}{\sin^2 \theta} + W.a \sin \theta + W.2a \sin \theta + \frac{W}{2}.2a \sin \theta = 0 \ ; \quad c = 2a \sin^3 \theta$$

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Example 12

Two equal uniform rods AB, AC each of length $2a$ and weight W are smoothly jointed at A. BD is a weightless bar of length a, smoothly jointed at B and fastened at D to a small smooth light ring sliding on AC. The system is in equilibrium in a vertical plane with ends B and C resting on a horizontal plane. Show that the magnitude of the reaction at A is equal to $\dfrac{W}{12}\left(3\sqrt{2}-\sqrt{6}\right)$. Also show that the magnitude of the reaction at A is equal to the stress on BD and it makes an angle 15° with the horizontal.

Find the point where the line of action meets BC.
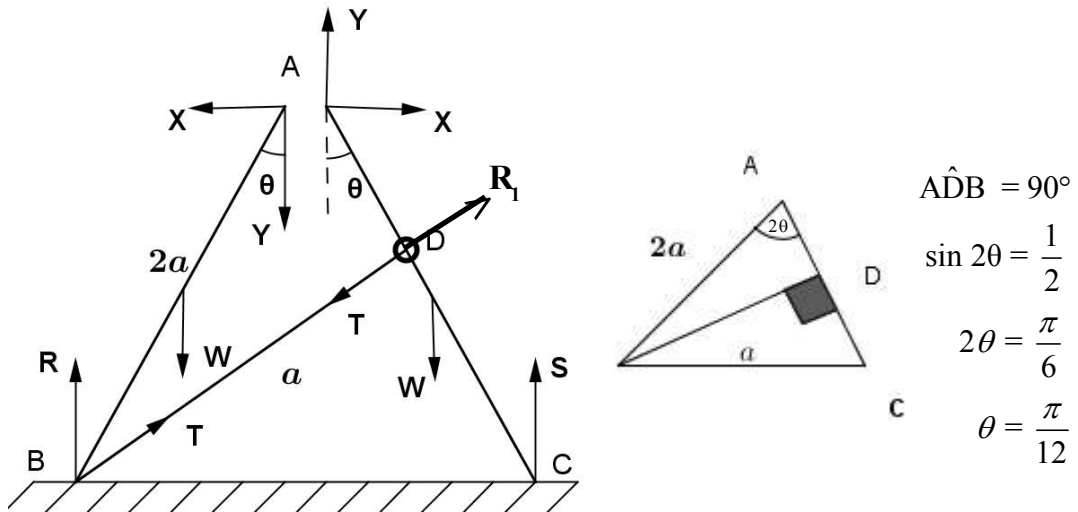
For the equilibrium of the ring $R_1 = T$ and R_1 is perpendicular to AC, so T is perpenticular to AC

For the system
Resolve the forces vertically

$\uparrow\ R + S = 2W$

By taking moment about C

Cm $\quad W.a \cos 75° + W. 3a \cos 75° = R. 4a \cos 75° = 0$

$$\Rightarrow R = W$$
$$R = S = W$$

Consider the equilibrium of the rod AC

By taking moments about A for AC

Am $\quad T. a\sqrt{3} + W.a \sin 15° - W.2a \sin 15° = 0$

$$T = \frac{W \sin 15°}{\sqrt{3}} = \frac{W}{12}\left(3\sqrt{2}-\sqrt{6}\right)$$
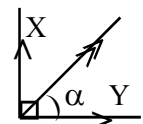
For the rod AB resolve the forces horizontally and vertically

$$A = \sqrt{X^2 + Y^2} = T \ ; \qquad \tan\alpha = \frac{X}{Y}$$

$\rightarrow\quad X = T \cos 15° \ ; \qquad \uparrow \qquad Y = T \sin 15° \ ;$

$$= \tan 15°$$

Using sine rule in $\triangle ABP \quad \dfrac{BP}{\sin 60°} = \dfrac{AB}{\sin 45°} \Rightarrow BP = \dfrac{2a . \sin 60°}{\sin 15°}$

$$\alpha = 15^0$$

$$BP = \frac{2a.\dfrac{\sqrt{3}}{2}}{\dfrac{\sqrt{3}-1}{2\sqrt{2}}} = \frac{2\sqrt{6}}{\sqrt{3}-1} \qquad \Rightarrow BP = \left(3\sqrt{2}+\sqrt{6}\right)a$$

14
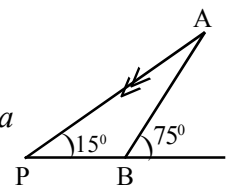
5.4 Exercises

1. Two uniform rods AB and AC of equal length are freely hinged at B. The weights of AB and BC are W_1 and W_2 respectively. The system freely hangs from fixed points B and C at the same level and $BC = 2a$. If the depth of A below BC is a, find the horizontal and vertical components of the reaction at A.

2. Two equal uniform rods AB, BC each of weight W are smoothly jointed at B and their midpoints are connected by an inextensible string. The string has length such as when it is taut $A\hat{B}C$ makes $90°$. The system is suspended from A freely while the string is taut. Show that the inclination of AB with the vertical at equilibrium is $\tan^{-1}\left(\dfrac{1}{3}\right)$ and the tension in the string is $\dfrac{3W}{\sqrt{5}}$. Also find the reaction on BC and show that it is in the direction of BC.

3. Two uniform equal rods AB, AC of length $2a$ and weight W, smoothly jointed at A lie symmetrically on the curved surface of a right circular cylinder whose axis is fixed horizontally. If each rod makes an angle θ with the horizontal and r is the radius of the cylinder, show that $r = a\operatorname{cosec}\theta\cos^3\theta$. Find also the reaction at A.

4. AB, BC and AC are three uniform equal rods smoothly jointed at ends A, B and C. AB and AC are each of weight W and the weight of BC is $2W$. The frame hangs freely from C. Show that BC makes an angle $\tan^{-1}\left(\dfrac{4}{\sqrt{3}}\right)$ with the horizontal. Find also the reaction at A and B.

5. Two uniform equal rods AOB and COD each of weight W are freely jointed at O, $AO = CO = a$, and $BO = OD = 3a$. At equilibrium B and D rest on a horizontal plane and B, D are connected by an inextensible string of length $3a$. The system lies in equilibrium in a vertical plane. Show that the tension in the string is $\dfrac{2\sqrt{3}\,W}{9}$ and find the reaction at O.

6. Two uniform equal rods AB and AC of weight $2W$ and W, respectively, are smoothly jointed at A. B and C are fixed to a horizontal log. Find the horizontal and vertical components of the reaction at A. If the reaction at B and C are perpendicular to each other and $A\hat{B}C = \alpha$, show that $3\cot\alpha = \sqrt{35}$.

7. Three uniform equal rods OA, AB and BC each of length $2a$ and weight W are freely jointed at A and B. The end O is hinged to a fixed point and a horizontal force P is applied to BC at C and BC makes an angle $45°$ to the horizontal. Find P in terms of W. Show that the reaction at O is $\dfrac{\sqrt{37}\,W}{2}$. Show also that C is at a horizontal distance $2a\left[\dfrac{1}{\sqrt{2}} + \dfrac{1}{\sqrt{10}} + \dfrac{1}{\sqrt{26}}\right]$ from the vertical through O.

8. Two equal uniform rods AB, BC each of length a and weight W are smoothly jointed at B. The rod AB is free to rotate about the point at which A is hinged. A small light ring is attached to C which is free to slide along another fixed rod through A. The fixed rod is inclined downwards, making an angle α to the horizontal. If the system is in equilibrium show that

    (i) $\tan B\hat{A}C = \dfrac{1}{2}\cot \alpha$

    (ii) The horizontal component of the reaction at B is $\dfrac{3W}{8}\sin 2\alpha$.

9. Four uniform equal rods AB, BC, CD and AD each of weight W are smoothly jointed at their ends to form a rhombus ABCD and hangs from A. The system is maintained in the shape of a square connecting the midpoints of BC and CD by a light rod. Find the thrust in the light rod and the reaction at C.

10. Five uniform equal rods AB, BC, CD, DE and EA each of weight W are freely jointed at their ends A, B, C, D and E to form a pentagon. The rods AB and AE make equal angles α and the rods BC and ED make equal angles β with the vertical. The system is hanged from A and the pentagon shape is maintained by connecting B and E by a light rod.

    (i)   Find the horizontal and vertical components of the reaction at C.

    (ii) Show that the stress in BE is $W(\tan \alpha + \tan \beta)$.

    (iii) Find the value of the stress when the pentagon is regular.

11. Four equal uniform rods AB, BC, CD and DA each of length $2a$ and weight W are smoothly jointed at A, B, C and D. The midpoints of BC and CD are connected by a light rod of length $2a\sin\theta$. The frame is freely hanged from A.

    (i)   Show that the thrust in the light rod is $4W\tan\theta$.

    (ii) Find the reaction at B and C.

12. Four equal uniform rods AB, BC, CD and AD each of weight W are smoothly jointed at their end points to make a square ABCD. The frame is hanged from A. The shape is maintained by joining the midpoints of AB and BC by an inextensible string.

    (i)   Show that the reaction at D is horizontal and its magnitude is $\dfrac{W}{2}$

    (ii) Show that the tension in the string is $4W$

    (iii) Show that the reaction at C is $\dfrac{W\sqrt{5}}{2}$ and it makes an angle $\tan^{-1}\left(\dfrac{1}{2}\right)$ with the vertical.

    (iv) Show that the reaction at B is $\dfrac{W\sqrt{17}}{2}$ and it makes an angle $\tan^{-1}\left(\dfrac{1}{4}\right)$

13. Four uniform equal rods AB, BC, CD and DA each of weight W are smoothly jointed at the ends to form a square ABCD. The frame is suspended from A and a weight $3W$ is attached to the point C. The shape is maintained by connecting the midpoints of AB and AD with a light rod. Show that the thrust in the light rod is $10W$.

14. Four uniform rods of equal length ℓ and weight W are freely jointed to form a framework ABCD. The joints A and C are connected by a light elastic string of natural length a. The framework is freely suspended from A and takes the shape of a square. Find the modulus of elasticity of the string. Find also the reaction at the joints B and D.

15. Six uniform equal rods each of weight W are smoothly jointed at their end points to form a hexagon ABCDEF. The system is suspended from A and the shape is maintained by light rods BF and CE. Show that the stress in BF is five times the stress in CE.

16. A uniform rod is cut into three parts AB, BC and CD of lengths ℓ, 2ℓ and ℓ respectively. They are smoothly jointed at B and C and rest on a fixed smooth sphere whose radius is 2ℓ and centre O, so that the middle point of BC and the extremities A and D are in contact with the sphere. Show that the reaction on the rod BC at its mid point is $\dfrac{91W}{100}$ where W is the weight of the rod.

    Find the magnitude and the direction on the rod CD at the joint C and the point whose line of action meets OD.

17. Three uniform rods AB, BC and AC of equal length a and weight W are freely jointed together to form a triangle ABC. The framework rests in a vertical plane on smooth supports at A and C so that AC is horizontal and B is above AC. A mass of weight W is attached to a point D on AB where

    $AD = \dfrac{a}{3}$. Find the reaction at joint B.

18. Two uniform equal rods AB and AC each of weight W and length $2a$ are freely jointed at A and placed in a vertical plane with ends B and C on a smooth horizontal table. Equilibrium is maintained by a light inextensible string which connects C to the mid point of AB with each rod making an angle

    $\alpha\left(<\dfrac{\pi}{2}\right)$ with the horizontal. Show that the tension T in the string is $T = \dfrac{W}{4}\sqrt{1+9\cot^2\alpha}$. Find the magnitude and the direction of the reaction at A.

19. Five uniform equal rods each of weight W are smoothly jointed at their ends to form a regular pentagon. CD is placed on a horizontal plane so that the frame is in a vertical plane and the shape is maintained by joining the midpoints of BC and DE by a light rod. Find the reaction at B and show that

    the tension in the light rod is $\left[\cot\dfrac{\pi}{5} + 3\cot\dfrac{2\pi}{5}\right]W$.

20. Three equal uniform rods AB, BC, CD each of length $2a$ and weight W are smoothly jointed at B and C, and rest with AB, CD in contact with two smooth pegs at the same level. In the position of equilibrium AB and CD are inclined at an angle α to the vertical BC being horizontal. Prove that the

    distance between the pegs is $2a\left(1+\dfrac{2}{3}\sin^3\alpha\right)$. If β is the angle which the reaction at B makes with the

    vertical, prove that $\tan\alpha \cdot \tan\beta = 3$.

6.0 Framework

In this chapter we will consider a framework consists of light rods joined at their ends to other rods with smooth joints.

6.1 Rigid Frame

If the shape of a frame is unaltered by external forces, then the frame in called a rigid frame.

In a frame made by light rods, the reactions at the joints will act along the rods. These reactions along the rods are known as stresses.

If we consider a light rod AB in a frame R_A and R_B are the reactions at the joints by pins. The rod is in equilibrium under the action of these two forces R_A and R_B. Hence for the equilibrium of the rods R_A and R_B must be equal and act opposite along the rod.

$R_A = R_B = T$

(i) T is tension

(ii) T is thrust
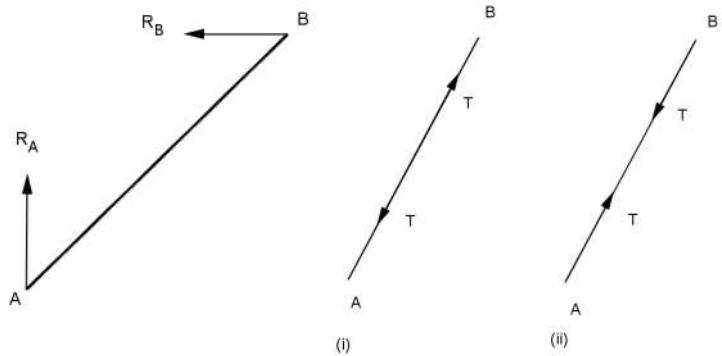
Assumptions when solving framework problems

● All the rods in the framework are light rods.

● All the rods are freely (smoothly) jointed at their ends and no couple in formed at a joint.

● The reactions at the joints (except external forces) will act along the rods. These may be thrusts or tensions.

● All the rods in the frame are in the same vertical plane and all the forces (including the external forces) are coplanar forces.
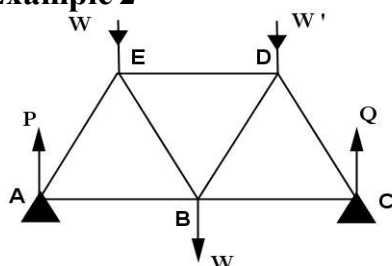
● External forces are applied only on joints.

6.2 Representing external forces in a light framework in equilibrium

Example 1

$\triangle$ ABC is frame lying on A and C, carries a load W at B. By symmetry the reactions at A and C are equal.
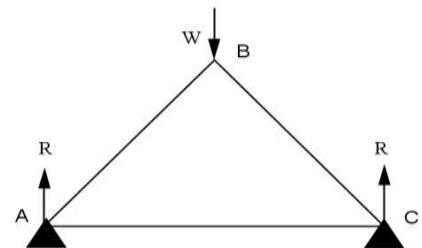
Example 2

ABCDE is a frame made of seven equal light rods and rests on two pegs at A and C. It carries W at E, B and W' at D. The external forces P, Q will be vertical.

Bow's Notation

- This notation is introduced by a mathematician called Bow.

- All the external forces will be represented outside of the frame.

- The region between forces (open or closed) is denoted by a small letter of the English alphabet or a number.

- Each force denoted by two letters of the alphabet belongs to the two region formed by the force.

Solving problems using Bow's notation

(i)   Having represented all external forces and regions, forces of polygons have to be drawn for each joint of the frame (These polygon of forces will be a closed figure, the vertices of the polygon being denoted by the names of the letters of the regions.)

(ii) The values of the stresses in rods can be calculated by using trigonometric ratios and algebraic equations in the triangles and polygons obtained in the stress diagram.

(iii) By reading the names of the sides in the stress diagram, mark the directions of the stress by using arrow marks.

(iv) While drawing force polygons, the disense has to be same for all the joints. (either clockwise or anticlockwise)

(v)   To draw a polygon of forces at a joint there may be maximum of two unknown forces.

6.3 Worked examples

Example 1
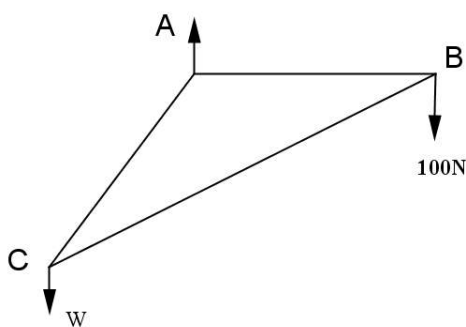
In the given figure, ABC is a triangular framework consisting of three smoothly jointed light rods AB, BC, CA, where AB = AC and BÂC=120° . The framework is in a vertical plane with AB horizontal. It is supported at A by a smooth peg and carries loads 100 N at B and W N at C. Draw a stress diagram using Bow's notation and from it, calculate the stresses in the rods, distinguishing between tensions and thrusts and also find the value of W.

Start from joint B

Joint	Order	Name of Polygon
B	$a \to b \to c \to a$	Δabc
C	$a \to c \to d \to a$	Δbcd
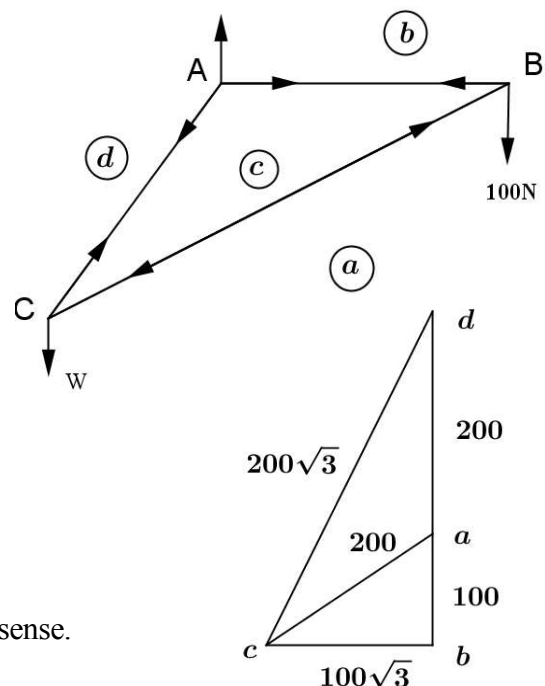
AB(bc) = Tension $= 100\sqrt{3}$ N

BC(ca) = Thrust $= 200\sqrt{3}$ N

CA(cd) = Tension $= 200\sqrt{3}$ N

W(ad) = 200 N

In this problem all the joints are taken in the anticlockwise sense.

Example 2

ABC is a frame obtained by joining three uniform equal light rods AB, BC and AC. B and C rest on 2 pegs at the same horizontal level. A carries a load of 100N. Find the reaction at B and C. Draw a stress diagram by using Bow's notation. Hence find the stress in each rod distinguishing between tension and thrust.
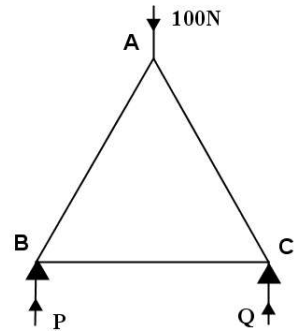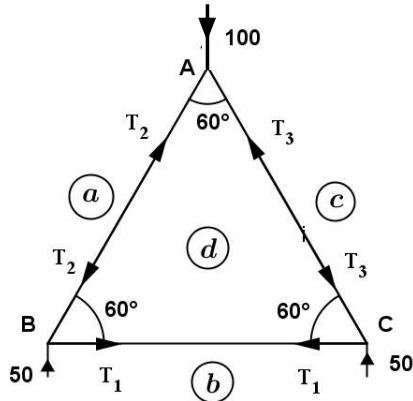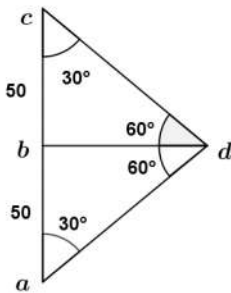

For equilibrium

Resolve the forces vertically

$$\uparrow P + Q = 100$$

$$P = Q = 50 \quad \text{(symmetry)}$$

Polygon of forces has to be drawn for joints A, B and C by naming the regions between the vertices as a, b, c and d.

Stress diagram

This diagram is drawn by taking the region around each joint in anticlockwise disense starting from C.

Joint C → Joint A → Joint B

Joint	order	Name of Polygon
C	$b \to c \to d \to a$	Δbcd
A	$d \to c \to a \to d$	Δacd

Tensions and thrusts are denoted by naming the regions.

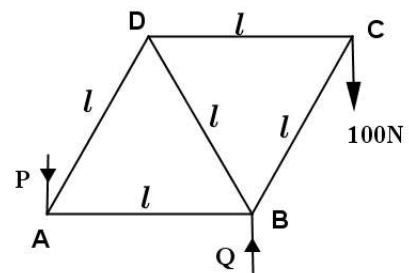$$T_1 = bd = 50 \tan 30° = \frac{50}{\sqrt{3}} \text{ N}$$

$$T_3 = cd = 50 \sec 30° = \frac{100}{\sqrt{3}} \text{ N}$$

$$T_2 = ad = 50 \sec 30° = \frac{100}{\sqrt{3}} \text{ N}$$

Rod	Stress	Thrust	Tension
AB	$\frac{100}{\sqrt{3}}$ N	✓	-
BC	$\frac{50}{\sqrt{3}}$ N	-	✓
AC	$\frac{100}{\sqrt{3}}$ N	✓	-

Example 3

The given figure represents the framework of five equal light rods. This frame is supported by a peg at B and a vertical force P is applied at A. C carries a load of 100 N. Find the stresses in each rod by drawing a stress diagram.
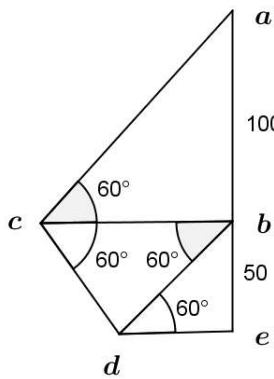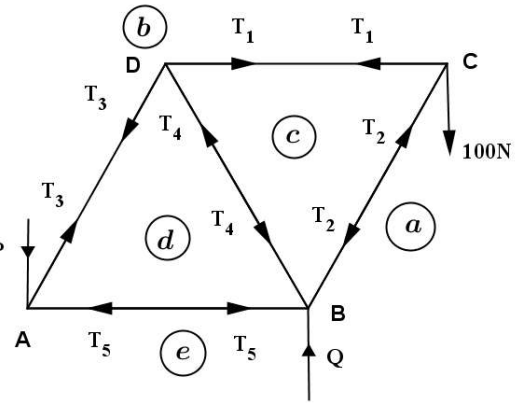
For equilibrium

Resolve the forces vertically

$\uparrow\ 100 + P = Q$ .... ①

By taking moments about A

Am $\quad Q.2\ell = 100\,(2\ell + 2\ell\cos 60°)$ ....② P

① and ② $\Rightarrow P = 50\ N$, $Q = 150\ N$

In the above diagram regions are named starting from C and the stress diagram is drawn as follows.

Joint	order	Name of Polygon
C	$a \to b \to c \to a$	Δabc
D	$c \to d \to c \to b$	Δbcd
A	$d \to b \to e \to d$	Δdbe
B	$c \to d \to e \to a \to c$	$\square acde$

The force polygon is drawn starting from joint C joining the region in the anticlockwise disense.

$T_1 = bc = 100 \tan 30° = \dfrac{100\sqrt{3}}{3}\ N$

$T_2 = ac = 100 \sec 30° = \dfrac{200\sqrt{3}}{3}\ N$

$T_3 = bd = 50 \csc 60° = \dfrac{100\sqrt{3}}{3}\ N$

$T_4 = cd = bd = \dfrac{100\sqrt{3}}{3}\ N$

$T_5 = dc = 50 \tan 30° = \dfrac{50\sqrt{3}}{3}\ N$

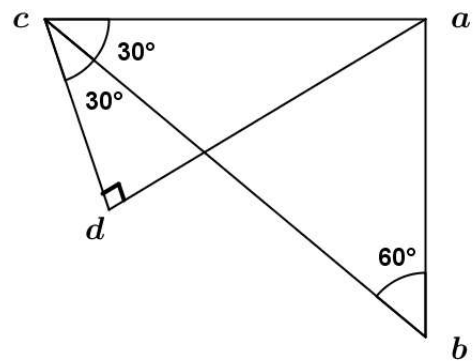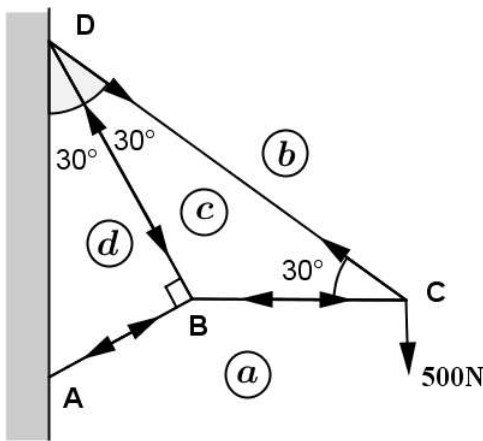| Rod	Stress	
DC	$\dfrac{100\sqrt{3}}{3}\ N$	Tension
BC	$\dfrac{200\sqrt{3}}{3}\ N$	Thrust
AD	$\dfrac{100\sqrt{3}}{3}\ N$	Tension
BD	$\dfrac{100\sqrt{3}}{3}\ N$	Thrust
AB	$\dfrac{50\sqrt{3}}{3}\ N$	Thrust

Example 4

A framework formed by four light rods AB, BC, CD and BD is shown in the given diagram. A, D are freely jointed to a vertical wall. Joint C carries a load of 500 N and BC remains horizontal. Draw a stress diagram using Bow's notation and find the stresses in each rod distinguishing between tensions and thrusts.
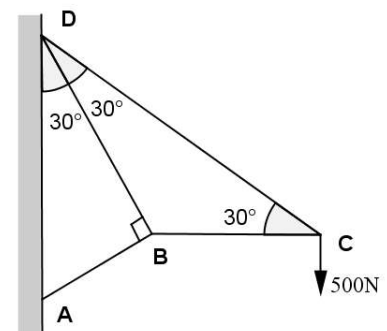
At joint C one force is known and two forces unknown. Draw the stress diagram starting from joint C.

Joint	order	Name of Polygon
C	$a \to b \to c \to a$	Δabc
B	$a \to c \to d \to a$	Δacd

bc = 500 sec 60° = 1 000 N

ac = 500 tan 60° = $500\sqrt{3}$ N

cd = ($500\sqrt{3}$ N) sin 30° = $250\sqrt{3}$ N

ad = $500\sqrt{3}$ N cos 30° = 750 N

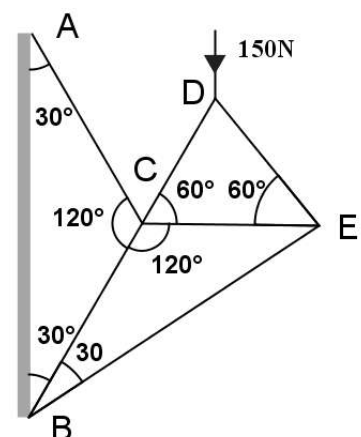| Rod	Stress	Thrust	Tension
DC	1 000 N	-	✓
BC	$500\sqrt{3}$ N	✓	-
BD	$250\sqrt{3}$ N	✓	-
AB	750 N	✓	-

Example 5

The given figure show a framework of six light rods smoothly jointed at C, D and E. A and B are smoothly jointed to a vertical wall and D carries a load of 150N. Draw a stress diagram using Bow's notation and find the stresses in each rod distinguishing between thrusts and tensions.

D is the joint with one known and two unknown forces

So start to draw triangle of forces from joint D

Joint	order	Name of Polygon
D	$p \to q \to r \to p$	Δpqr
E	$p \to r \to s \to p$	Δprs
C	$r \to q \to t \to s \to r$	$\Box rqts$
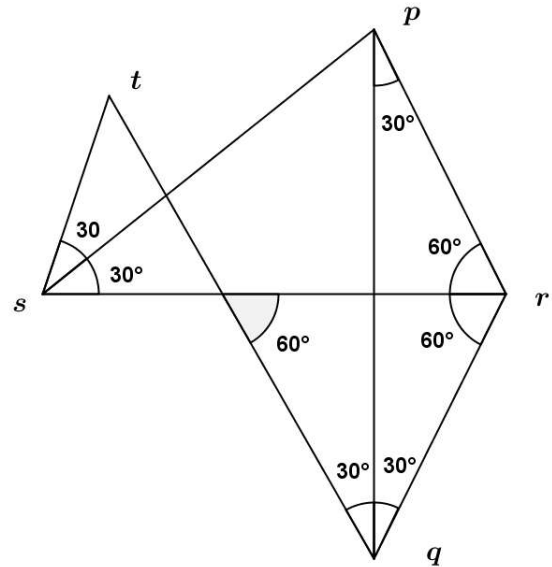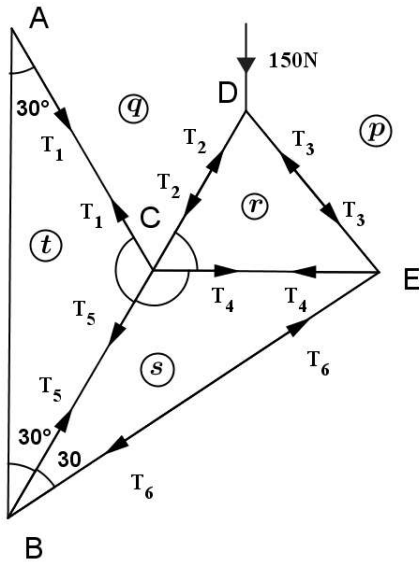
$$AC = qt = 75\sqrt{3} + 25\sqrt{3} = 100\sqrt{3} \text{ N}$$

$$CD = qr = 75 \sec 30° = 50\sqrt{3} \text{ N}$$

$$DE = pr = qr = 50\sqrt{3} \text{ N}$$

$$CE = sr = 100\sqrt{3} \text{ N}$$

$$BC = st = 50\sqrt{3} \text{ N}$$

$$BE = ps = 150\sqrt{3} \text{ N}$$

Rod	Stress	Thrust	Tension
AC	$100\sqrt{3}$ N	-	✓
CD	$50\sqrt{3}$ N	✓	-
DE	$50\sqrt{3}$ N	✓	-
CE	$100\sqrt{3}$ N	-	✓
BC	$50\sqrt{3}$ N	-	✓
CE	$150\sqrt{3}$ N	-	✓

Example 6

Five rods AB, BC, CD, DA and AC are smoothly jointed at their ends to form a framework as shown in the figure. $A\hat{B}C = A\hat{D}C = D\hat{A}C = 30°$ and $B\hat{A}C = 60°$. The framework is smoothly hinged at D and carries a weight $10\sqrt{3}$ N at B. The framework is held in a vertical plane with AB horizontal by a horizontal force P at A.

(i)   Find the value of P

(ii) Find the magnitude and direction of the reaction at D.

(iii) Using Bow's notation, draw a stress diagram for the framework and find the stresses in all the rods, distinguishing between tensions and thrusts.
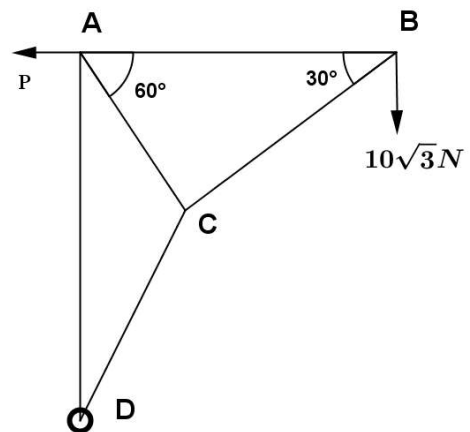
(i)   For equilibrium

   Take moment about D

$$D\hat{m} \qquad P. AD - 10\sqrt{3} \text{ AB} = 0$$

$$\text{but} \quad AD = 2\,AC\cos 30°$$
$$= 2\,AB\cos 60°\cos 30°$$
$$AD = \frac{\sqrt{3}}{2}AB$$
$$\therefore \quad P.\frac{\sqrt{3}}{2}AB = 10\sqrt{3}\,AB$$
$$\therefore \quad P = 20\,N$$

Let R be the reaction at D and θ be the angle that R makes with the horizontal
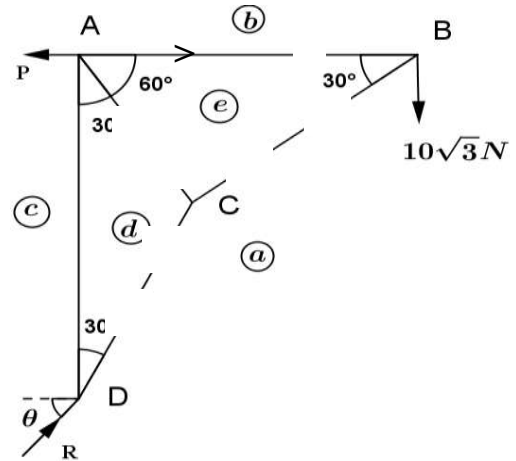
Resolving vertically

$$\uparrow R\sin\theta = 10\sqrt{3}$$

Resolving horizontally

$$\rightarrow R\cos\theta = P = 20\,N$$

$$R = \sqrt{\left(10\sqrt{3}\right)^2 + 20^2} = 10\sqrt{7}$$

$$\tan\theta = \frac{10\sqrt{3}}{20} = \frac{\sqrt{3}}{2}\;;\qquad \theta = \tan^{-1}\frac{\sqrt{3}}{2}$$

Since the system is in equilibrium under three forces the reaction R should also pass through B.

Start from joint B in the anticlockwise direction.

Joint	order	Name of Polygon
B	$a \rightarrow b \rightarrow e \rightarrow a$	Δabd
C	$a \rightarrow e \rightarrow d \rightarrow a$	Δaed

Rod	Stress	Magnitude
AB	Tension	30 N
BC	Thrust	$20\sqrt{3}$ N
AC	Thrust	20 N
DC	Thrust	40 N
AD	Tension	$10\sqrt{3}$ N

$$be = 10\sqrt{3}\tan 60$$
$$= 30$$
$$ae = 10\sqrt{3}\sec 60$$
$$= 20\sqrt{3}$$
$$ad = 20\sqrt{3}\sec 30$$
$$= 40$$
$$ed = ae\tan 30$$
$$= 20$$
$$cd = ed\sin 60$$
$$= 10\sqrt{3}$$
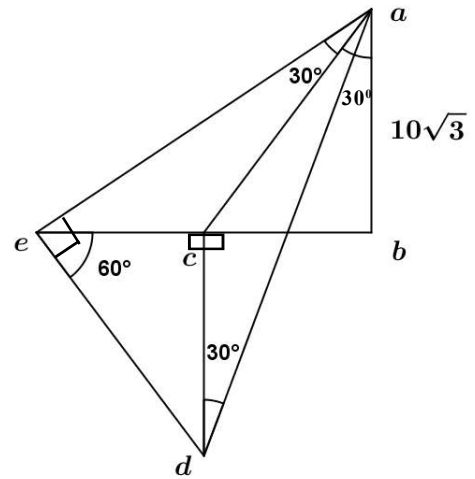
Example 7
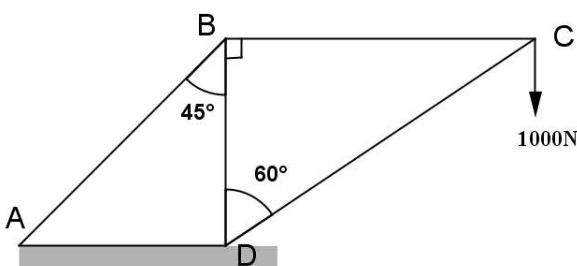
The given figure shows a crane composed of four freely jointed rods AB, BC, CD and BD. The rod BC is horizontal while the rod BD is vertical. The crane is fixed to the horizontal ground at A and D and there is a load of 1 000 N hanging at C. Use Bow's notation to find the forces in the rods, distinguishing between tensions and thrusts.

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Start with joint C in anticlockwise

Joint	order	Name of Polygon
C	$1 \to 2 \to 3 \to 1$	$\Delta 123$
B	$3 \to 2 \to 4 \to 3$	$\Delta 324$

$AB = km = 1000\sqrt{6}$ N

$BC = kl = 1\,000 \cot 30^0 = 1\,000\sqrt{3}$ N

$CD = lj = 1\,000 \csc 30° = 2\,000$ N

$BD = ml = kl = P.\ell \cos 30° - 10\sqrt{3} . 2\ell = 0$

$\Rightarrow P = 40$ N

$\to P = R \cos \theta = 40$ N

$\uparrow R \sin \theta = 10\sqrt{3}$ N

$R = \sqrt{40^2 + \left(10\sqrt{3}\right)^2}$ N

$R = 10\sqrt{19}$ N

Rod	Stress	Thrust	Tension
AB	$1000\sqrt{6}$ N	-	✓
BC	$1000\sqrt{3}$ N	-	✓
CD	$2\,000$ N	✓	-
BD	$1000\sqrt{3}$ N	✓	-
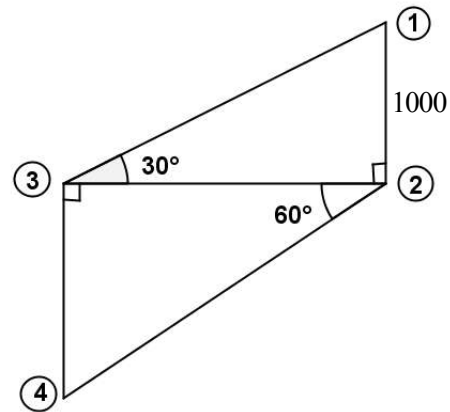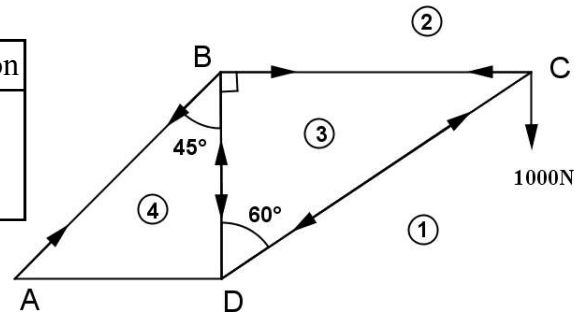
Example 8
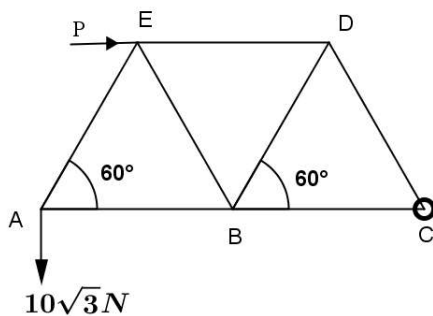
The given figure shows a framework consisting of seven light rods AB, BC, CD, DE, EA, EB and BD smoothly jointed at their extremities. The frame smoothly jointed at C and carries a load $10\sqrt{3}$ N at A. A horizontal force P at E keeps ED horizontal and the frame is in a vertical plane.

(i) Find the value of P at E

(ii) Find the magnitude and the direction of the reaction at C

(iii) Draw a stress diagram using Bow's notation and hence find the stresses in each rod distinguishing between tensions and thrusts.

(iv) From stres diagram varify reaction at C

For equilibrium

Take moment about C

$\cap P.\ell \cos 30^0 - 10\sqrt{3}.2\ell = 0$, where ℓ is length of a rod.

$\Rightarrow P = 40N$

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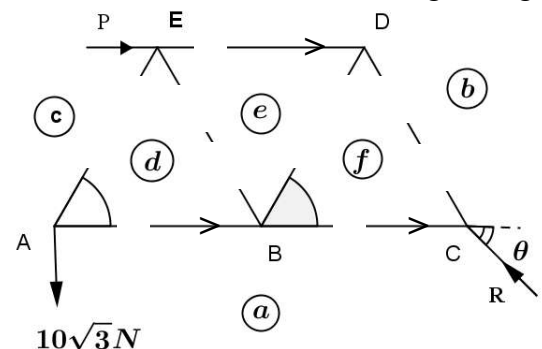Resolve the forces in the horizontal direction

$$\rightarrow \ P = R\cos\theta = 40N$$

Resolve vertically
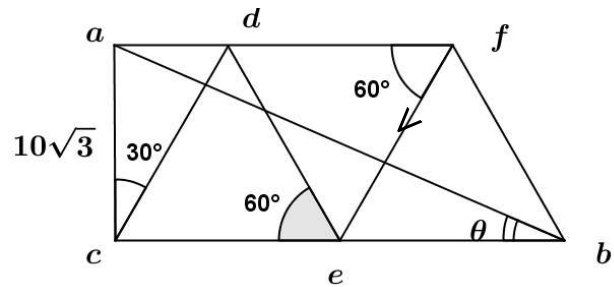
$$\uparrow R\sin\theta = 10\sqrt{3}N$$

$$R = \sqrt{40^2 + \left(10\sqrt{3}\right)^2}$$

$$R = 10\sqrt{19}N$$

$$\tan\theta = \frac{10\sqrt{3}}{40} \ \Rightarrow \ \theta = \tan^{-1}\left(\frac{\sqrt{3}}{4}\right)$$

Stress diagram

Start from joint A clockwise

Joint	order	Name of Polygon
A	$c \rightarrow a \rightarrow d \rightarrow c$	Δcad
E	$c \rightarrow d \rightarrow e \rightarrow b \rightarrow c$	$\square cdeb$
D	$b \rightarrow e \rightarrow f \rightarrow b$	Δbef

Join af, bf

$AB = ad = 10\sqrt{3}\tan 30° = 10$ N

$AE = cd = 10\sqrt{3}\sec 30° = 20$ N

$cd = de = 20$ N

$AE = BE = 20$ N

$bf = de = ef = df = 20$ N

$CD = DE = BD = 20$ N

Reaction at c denoted by ab
$$ab^2 = \left(10\sqrt{3}\right)^2 + 40^2$$
$$ab = 10\sqrt{19}$$

Rod	Stress	
AB	10 N	Thrust
BC	30 N	Thrust
CD	20 N	Thrust
DE	20 N	Thrust
EA	20 N	Tension
EB	20 N	Thrust
DB	20 N	Tension

Example 9
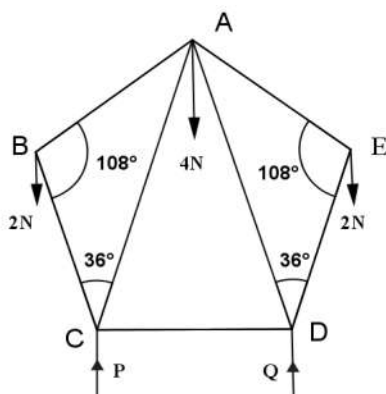
A framework of seven freely jointed light rods is in the form of a regular pentagon ABCDE and the diagonals AC and BD. The framework is in the vertical plane with the lowest rod CD horizontal and is supported at C and D by two upward vertical forces of magnitude P and Q and weights 2 N, 4 N, 2 N are suspended at B, A and E respectively. Draw a stress diagram for this framework using Bow's notation. Hence determine the stresses in all seven rods, distinguishing between tensions and thrusts. Give the answers in terms of $\cos\dfrac{n\pi}{10}$ where n is a positive integer.

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For the equilibrium of the system

  Resolve the forces vertically

$$P + Q = 8 \text{ N}$$

$\uparrow$ $\quad P = a$ $\qquad$ (symmetry)

$\quad P = Q = 4 \text{ N}$

The system is symmetrical about the vertical line through A.
Start from the joint B and move in the clockwise direction.

$$n = 18° \ (\text{say}) = \frac{\pi}{10}$$

$$de = ec = ca = ab = 2 \text{ N}$$

let $gc = x$

Then $pc = x \tan 4n°$

AB(bc) = Tension $\qquad = 100\sqrt{3} \text{ N}$

BC(ca) = Thrust $\qquad = 200\sqrt{3} \text{ N}$

CA(cd) = Tension $\qquad = 200\sqrt{3} \text{ N}$

W(ad) = 200 N
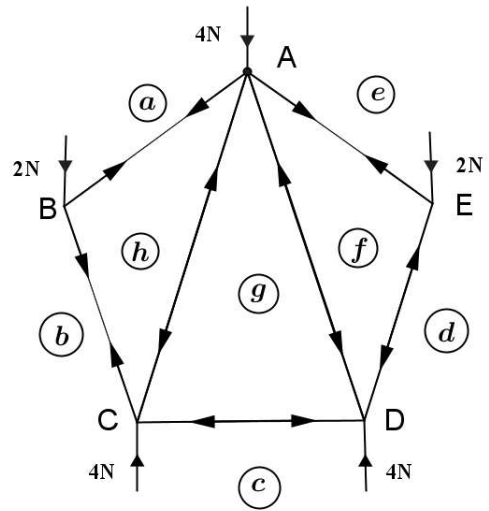
First draw a vertical line and denote the vertical forces, in clockwise sense as

  $\_$ ba, ae, ed, dc, ca

$$\theta = \frac{\pi}{10} = 18^0$$

Joint	Order	Polygon
B	b→a→h→b	△ bah
E	e→d→f→e	△ edf
A	h→a→e→f→g→h	⬠ haefg
C	b→h→a→c→h	▱ bhgc

In △abh , Using sine rule

$$\frac{ah}{\sin \theta} = \frac{2}{\sin 4\theta} = \frac{bh}{\sin 3\theta}$$

$$ah = 2\frac{\sin \theta}{\sin 4\theta} ,$$

$$bh = 2\frac{\sin 3\theta}{\sin 4\theta}$$

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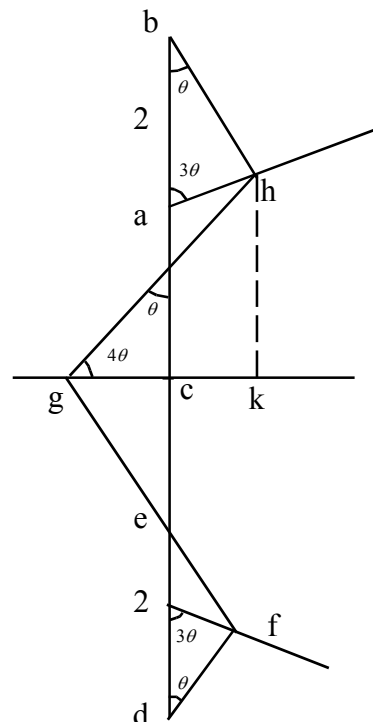
Δ 's abh and def are concruent

$$\therefore af = ah = \frac{2\sin\theta}{\sin 4\theta}$$

$$\therefore df = bh = \frac{2\sin 3\theta}{\sin 4\theta}$$

In Δ ghk

$$hk = gh\sin 4\theta = ac + ah\cos 3\theta$$

$$gh = \frac{2}{\sin 4\theta} + 2\frac{\sin\theta}{\sin 4\theta}.\cos 3\theta$$

$$gc + ah\sin 3\theta = gh\cos 4\theta$$

$$gc = gh\cos 4\theta - ah\sin 3\theta$$

Rod	stress	Tension	Thrust
AB	ah	✓	-
BC	bh	-	✓
AE	ah	✓	-
ED	bh	-	✓
AC	gh	-	✓
AD	gh	-	✓
DC	gc	✓	-

Example 10

The given frame consists of five light rods AB, AD, BC, BD and CD smoothly jointed at their extremities while the frame carrie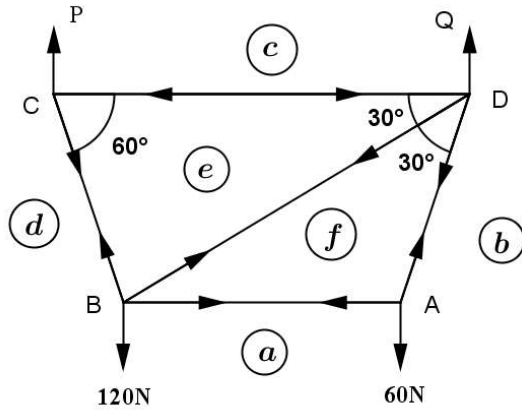s 120 N and 60 N at B and A respectively, the vertical forces P Newton and Q Newton applied at C and D to make AB and CD horizontal. Draw a stress diagram using Bow's notation and hence find the stresses in all five rods distinguishing between thrusts and tensions.

For equilibrium

Resolving vertically

$$\uparrow P + Q - 120 - 60 = 0$$

$$P + Q = 180N$$

Taking moments about D

$$P. 2\ell - 60.\ell \cos 60 - 120.\ell + \ell \cos 60 = 0$$

$$2P = 30 + 180$$

$$\Rightarrow P = 105N$$

$$\Rightarrow Q = 180 - 105 = 75N$$

Stress diagram

Start from joint C and move anticlockwise

$$C \quad \rightarrow \quad B \quad \rightarrow \quad A$$

Step I : Cm   Step II : Bm   Step III : Am
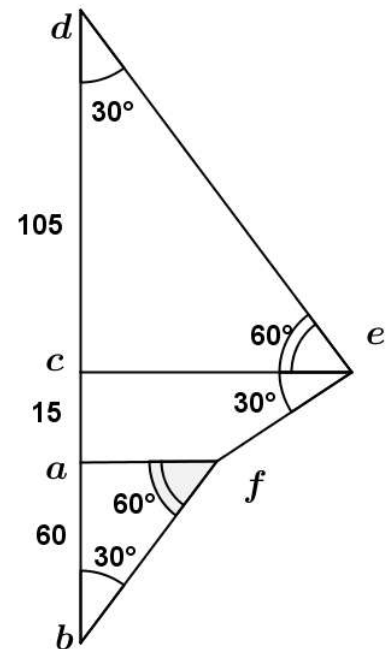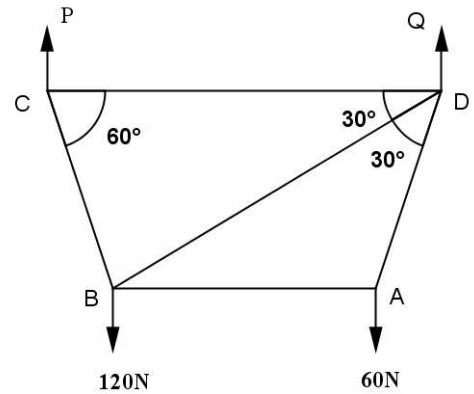
$$AB = af = 60 \tan 30° = 20\sqrt{3} \ N$$

$$BC = ed = 105 \sec 30° = 70\sqrt{3} \ N$$

$$CD = ec = 105 \tan 30° = 35\sqrt{3} \ N$$

$$AD = bf = 60 \sec 30° = 40\sqrt{3} \ N$$

$$BD = ef = 15 \sec 30° = 10\sqrt{3} \ N$$

Rod	Stress	
AB	$20\sqrt{3}$ N	Tension
BC	$70\sqrt{3}$ N	Tension
CD	$35\sqrt{3}$ N	Thrust
AD	$40\sqrt{3}$ N	Tension
BD	$10\sqrt{3}$ N	Tension

6.4 Exercises

1.
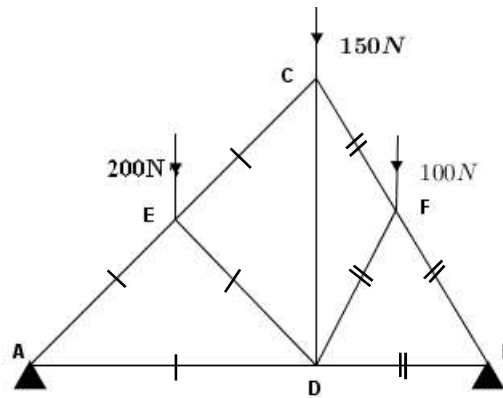
The figure represents the framework of a roof whose weight may be regarded as distributed in the manner shown above.

   i.   Find the reaction at A and B.

   ii.   Draw the stress diagram by using Bow's notation and find the stress in each rod, distinguishing between tensions and thrusts.

2.
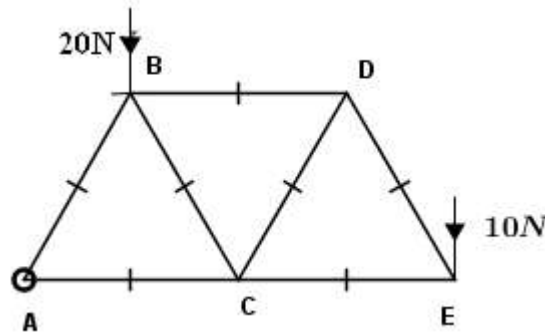
The above figure shows a framework made by seven light rods. The frame is hinged at A to a fixed point and kept in position by a horizontal force P at B. Draw a stress diagram using Bow's notation and find the stress in each rod distinguishing between tensions and thrusts.

3.
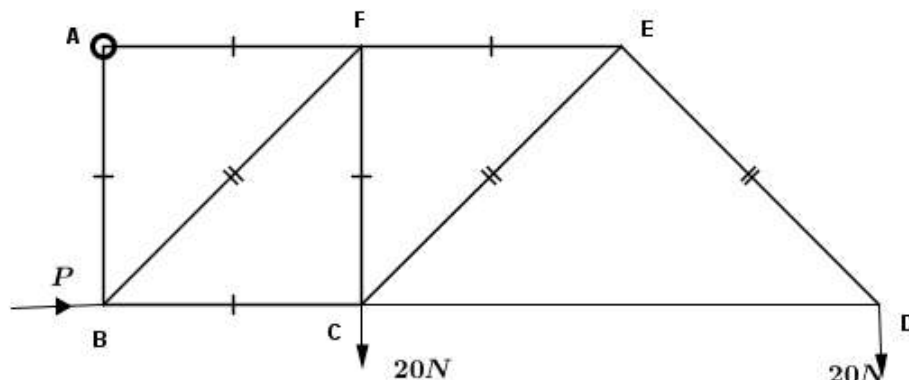
The above framework consists of nine smoothly jointed light rods, smoothly hinged to a fixed point at A, kept in equilibrium by a horizontal force P at B and loaded with 20 W each at C and D.

   i.   Find P and the reaction at A.

   ii.   Draw the stress diagram using Bow's notation and find the stress in each rod, distinguishing between tensions and compressions.

4. The framework consists of four light rods AB, BC, CD and DB freely jointed at B, C, D and attached to a vertical wall at A, and D loaded with WN at C. Draw a stress diagram using Bow's notation and find the stress in each rod, distinguishing between tensions and thrusts.

5.

ABCDEF is a framework which has freely jointed rods AB, BC, CD, DE, AE, BE and CE such that
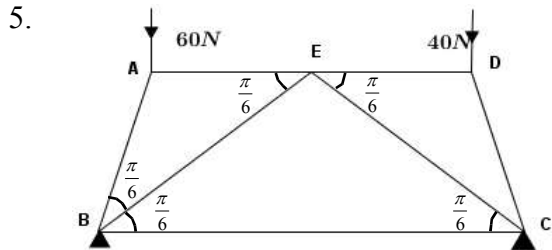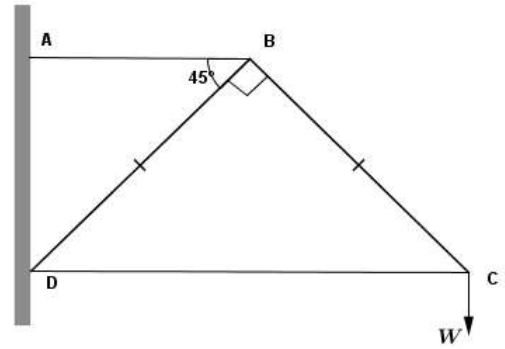
$$E\hat{B}C=E\hat{C}B=A\hat{B}E=D\hat{C}E=A\hat{E}B=D\hat{E}C=\frac{\pi}{6}.$$ The

framework is supported at B and C such that BC is horizontal, and loaded 60 N, 40 N at A and D respectively. Draw a stress diagram using Bow's notation and find the stress in each rod, distinguishing thrusts and tensions.

6. The framework in the figure is formed by using light bars according to the diagram. All triangles are right angular and isosceles. The system is on support at A and B such that ACB horizontal. The framework carries loads of 40 N, 400 N and 240 N at C, D and E respectively. Draw stress diagram using Bow's notation and find the stresses in the bars distinguishing tensions and thrusts.

7.
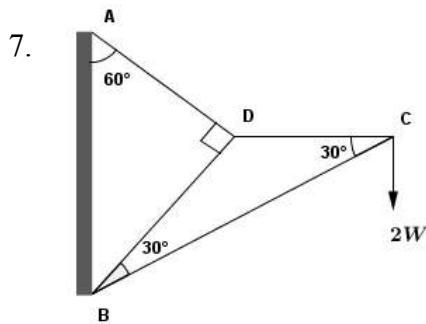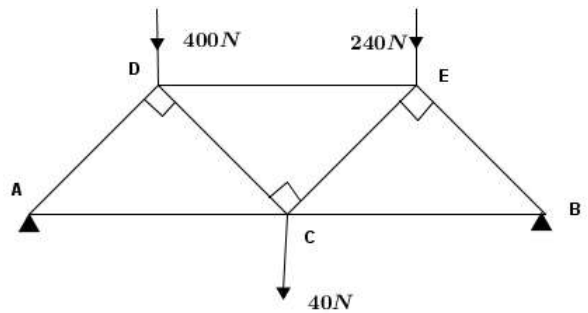
The figure shows a framework consisting freely jointed four light bars AD, BD, BC and CD. It is hinged freely to a vertical wall at A and B. C carries a load of $2W$. By using stress diagram find the reactions at A and B. Hence find the stresses in the rods distinguishing tensions and thrusts.

8. The figure shows a framework consisting nine light rods freely jointed at A, B, C, D, E and F. The frame carries loads of $6W$ and $9W$ at B and C respectively. It is supported by vertical forces R and S at A and D respectively. Draw a stress diagram and find the stresses in the rods distinguishing between tensions and thrusts.

9. A freely jointed framework consisting of five light rods is shown in the figure. Joint B carries a load of 900N. The framework is in equilibrium such as AD is vertical by means of forces P and (P, Q) acting on A and D respectively (P is horizontal and Q is vertical). Find the magnitudes of forces P and Q. Draw the stress diagram using Bow's notation and find the stress in each rod, distinguishing tensions and thrusts.

10.

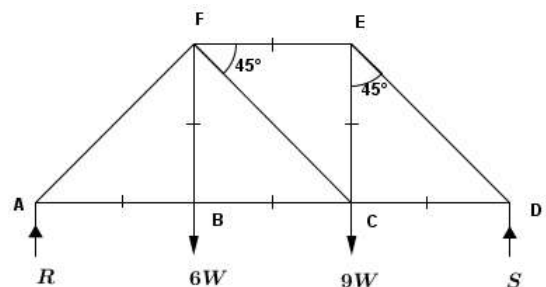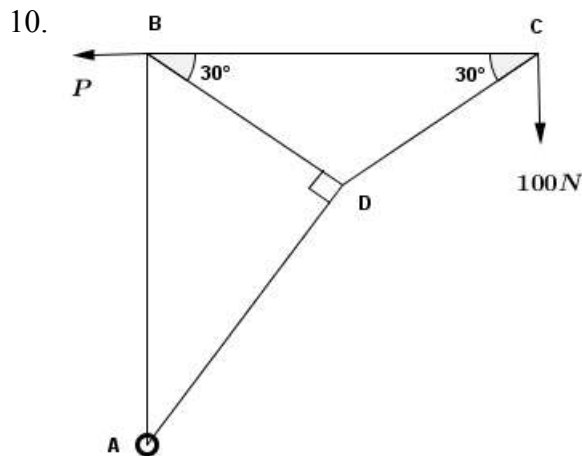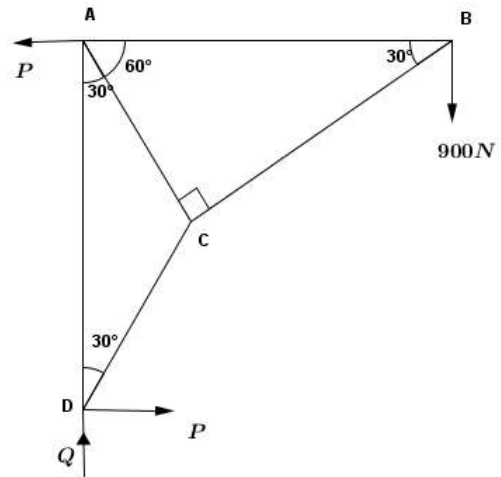Five light rods are freely jointed to form the framework shown in the above figure. The framework is in equilibrium in a vertical plane with joint A freely hinged to a fixed point. AB is vertical, BC is horizontal, $A\hat{D}B = 90°$ and $D\hat{B}C = D\hat{C}B = 30°$. A load of 100 N hangs at C and a horizontal force P acts at B in the direction of CB.

Find P and obtain the horizontal and vertical components of the reaction on the hinge at A. Draw a stress diagram for the framework using Bow's notation. Hence determine the stresses in all five rods distinguishing tensions and thrusts.

11. The given framework consists freely jointed eight light rods at A, B, C, D and E. The joints A and B are on vertical supports P at each joint. The framework carries equal loads of 100 kg at points C and D. AB is horizontal and AE=BE=AD=BC. Find the value of P. Assuming the thrust in C as X kg draw a stress diagram for the framework. If the tension on AB is Y kg, using the geometry of the stress diagram, prove that

$y = 100 - (\sqrt{3} - 1)X$. Explain why the real values of x and y cannot be calculated simultaneously. Find the stress in every rod if X = Y.
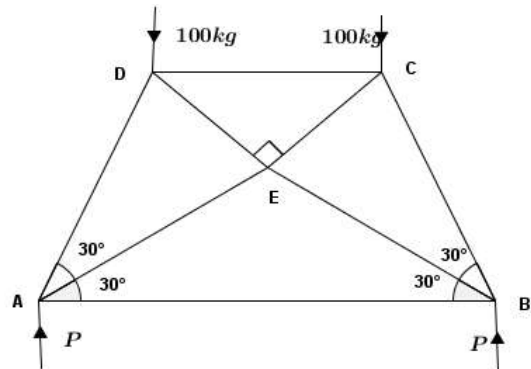
12. The given figure represents a framework which is formed by seven light rods. Ends A, B, C, D, E are freely jointed. This framework carries loads W and $2W$ at joints C and D, and is supported at B and E such that BE is horizontal. Draw a stress diagram using Bow's notation and find the stress in every rod distinguishing tensions and thrusts.

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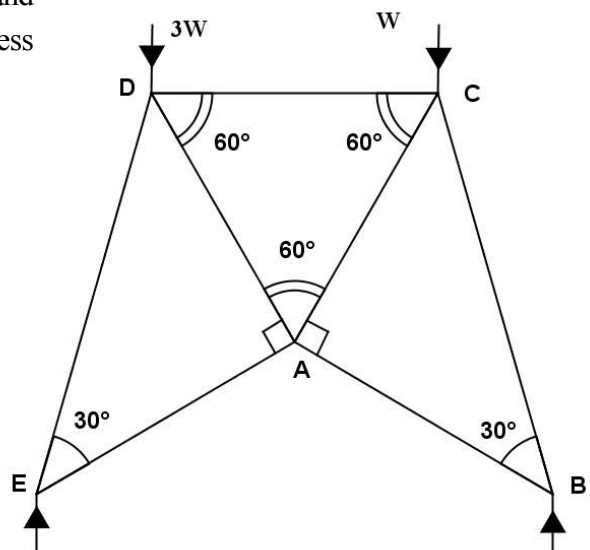
13. The framework consisting seven freely jointed light rods is placed on two supports at A and C such as the framework carries loads $4W$ and W at D and E respectively. Find the reactions at A and C. Find the stresses in each and every rod using a stress diagram, distinguishing between them compressions and tensions.
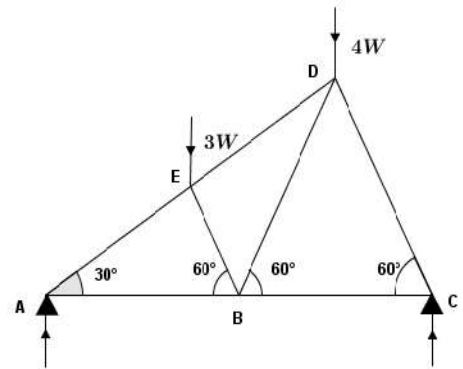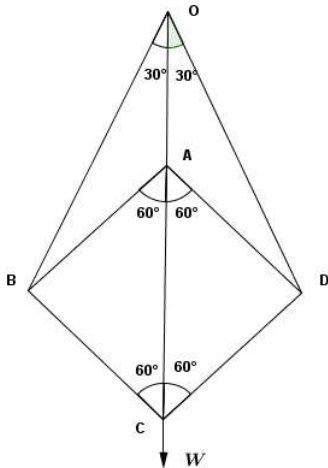
14. The given framework is a rhombus formed by freely jointed five light rods. It is hung from A by means of equal strings OB, OD, and OA is a vertical rod freely jointed at A. The diagonal AC of the rhombus is vertical and $A\hat{B}C = B\hat{O}D = 60°$. When C carries a load W, find the stresses in each rod and also the tensions in the strings by using a stress diagram. Name the rods which are under tension.

15. The figure shows a framework formed by freely jointed light bars. DA is vertical. The framework is supported at C and E. It carries loads $3W$, $3W$ and W at joints A, B and F, respectively. Find the reactions at C and E. Draw a stress diagram using Bow's notation and find the stresses in each rod. Distinguish thrusts and tensions.
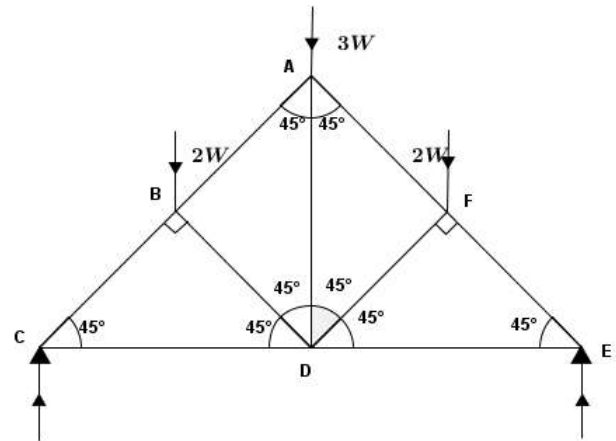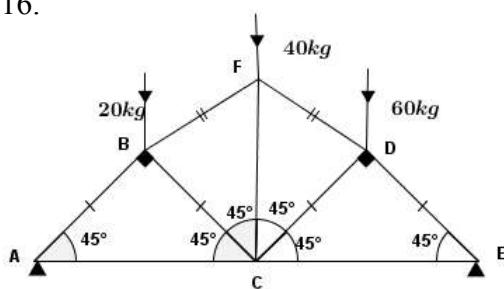
16. The given figure represents a framework of light bars loaded at joints B, F, D as indicated. The bars AC and CE are horizontal and each equal to 10 m and CF = 8 m. Also the lengths AB = BC = CD = DE and BF = FD. The frame rests on two smooth pegs at A and E. Calculate the reactions at A and E assuming that they are vertical. Draw a stress and find the stresses in the rods distinguishing between tensions and thrusts.

17. The given framework formed by nine equal light bars, carries loads as shown in the figure. The framework is at rest on B and C on supports such that the system is in a vertical plane.

Find the reaction at B and C. Draw a stress diagram by using Bow's notation. Hence calculate the stresses in each rod distinguishing tensions and thrusts.

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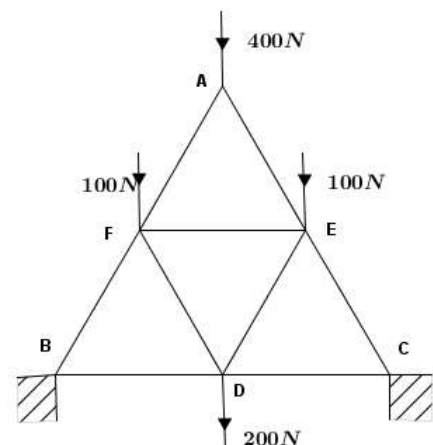
18. The framework of a bridge ABCDE which is formed by seven light equal rods is shown in the figure. The joints A and C are on supports which are in same horizontal level and the framework is in a vertical plane and B carries a load W. Draw a stress diagram using Bow's notation. Hence find the stress in each rod distinguishing thrusts and tensions.
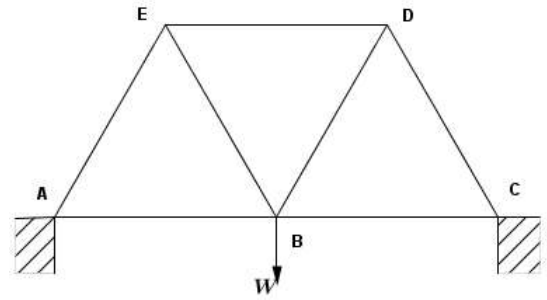
19.
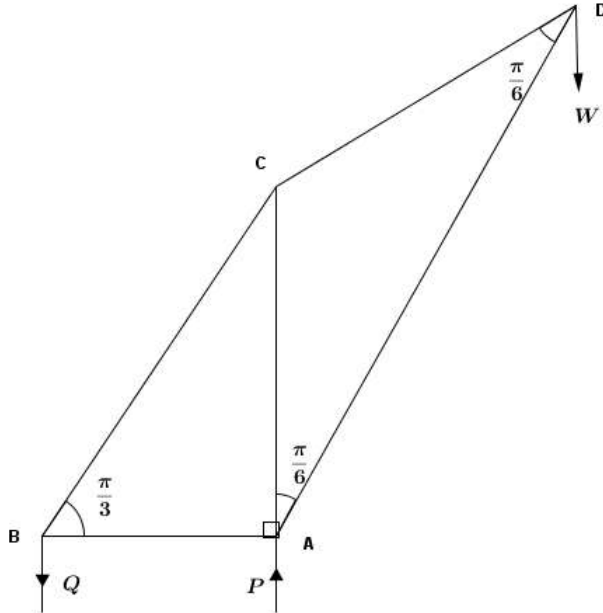
The framework consisting light rods AB, BC, CA, CD and DA are freely jointed at their extremities is placed in a vertical plane with AB horizontal and AC vertical, AB = a,

$$B\hat{C}D = B\hat{A}D = \frac{2\pi}{3} \text{ and } A\hat{B}C = \frac{\pi}{3}.$$ The framework supports a vertical load W at D and the equilibrium is maintained by two vertical forces P, Q at A and B respectively.

(i) Find P and Q in terms of W

(ii) Draw a stress diagram for this framework using Bow's notation.

Hence determine the stresses in the five bars distinguishing thrusts and tensions.

20.
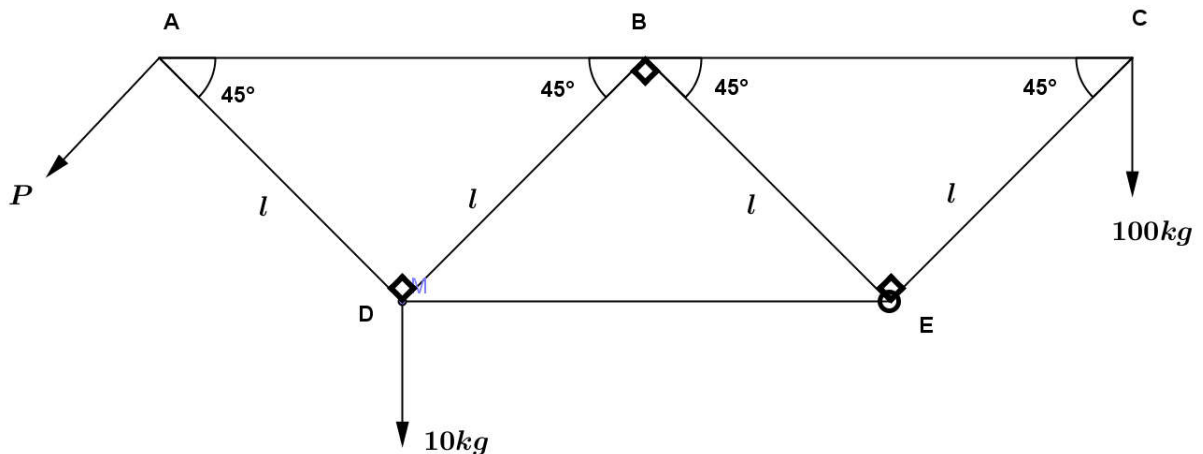
The above framework is made by seven light rods AB, BC, AD, BD, BE, CE and DE where AD = BD = BE = CE = ℓ. The frame is hinged at E, kept in equilibrium by a force P applied at A, with loads 100 kg at C and 10 kg at D.

(i) Find the vertical and horizontal components of the reaction at E.

(ii) Find the value of P.

(iii) Draw a stress diagram using Bow's notation and hence find stresses in each rod distinguishing tensions and thrusts.

7.0 Friction

7.1 Introduction

When two bodies are in contact with each other the action at the point of contact of the bodies to prevent sliding of one on another is called frictional force. The frictional force on two bodies are equal in magnitude and opposite in direction.

When a horizontal force P is applied on a body, if it does not move the reason is that the force P is suppressed by an equal and opposite force. This force is called frictional force and if this force is F, then
$F = P$.
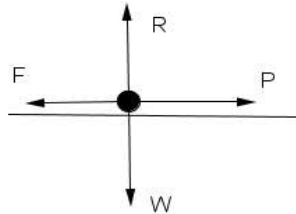
When P is gradually increased, at some stage the body will start to move. This shows that the frictional force cannot increase beyond a limit and this is called limiting frictional force.

At limiting equilibrium,

$$\text{Coefficient of friction} = \frac{\text{Limiting frictional force}}{\text{Normal Reaction}} = \mu \text{, where } \mu \text{ is the coefficient of friction}$$

In equilibrium $\dfrac{F}{R} \leq \mu$

At limiting equilibrium $\dfrac{F_L}{R} = \mu$. (F_L - Limiting frictional force)

7.2 Laws of Friction

1. When two bodies are in contact with each other, the direction of the frictional force at the point of contact acting on the body by the other is opposite to the direction in which the body tends to move.

2. When the bodies are in equilibrium the magnitude of the frictional force is sufficient only to prevent the motion of the body. Only a certain amount of friction can be exerted called limiting friction.

3. The ratio of the limiting frictional force and the normal reaction is called the coefficient of friction and depends on the matter of which the body is composed.

4. Until the normal reaction remains unchanged, the limiting frictional force does not depend on the area and the shape of the surfaces.

5. When the motion is started, the direction of the frictional force is opposite to the direction of the motion. The frictional force after the motion is started is slightly less than the limiting frictional force before the motion.

6. The frictional force exerted by the surface on a moving body does not depend on the velocity of the body.

Angle of Friction

When two bodies are in contact with each other the total reaction at the point of contact is the resultant of the normal reaction and the frictional force. At limiting equilibrium, the angle λ which this resultant makes with the normal reaction is called the angle of friction.

$$\tan\lambda = \frac{F_L}{R}$$

$$\frac{F_L}{R} = \mu$$

$$\tan\lambda = \mu$$
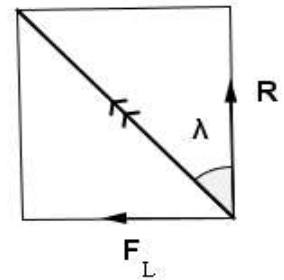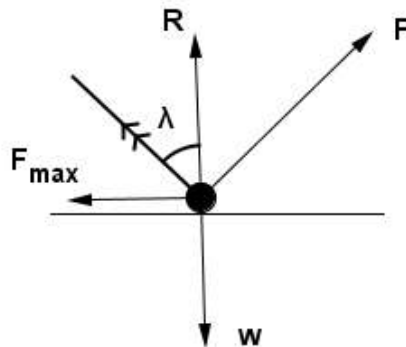
Cone of Friction
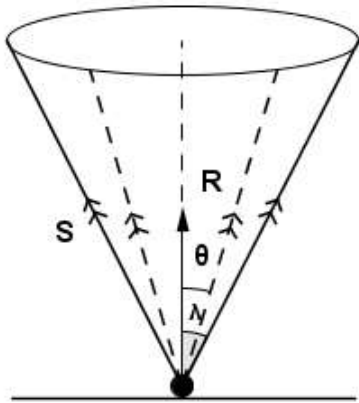
When a body is in contact with a rough surface and with the common normal at the point of contact as axes, we describe a right circular cone whose semi vertical angle is λ.

This cone is defined as cone of friction. The resultant reaction must always be within or on the surface of the cone whatever the direction the body tends to move.

- **Equilibrium of a particle on a rough horizontal surface when an external force acts**

$$\frac{F}{R} = \tan\theta$$

$$\frac{F}{R} \leq \mu$$

$$\tan\theta \leq \mu$$

$$\tan\theta \leq \tan\lambda$$

$$\theta \leq \lambda$$
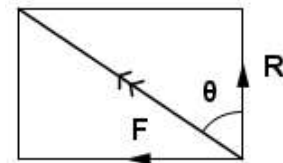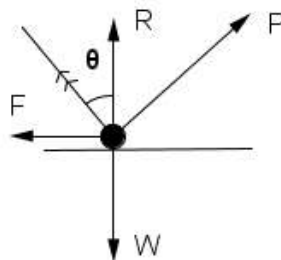
- **Equilibrium of a particle on a rough inclined plane**

Resolving parallel to the plane

$$\nearrow \quad F - W\sin\alpha = 0 \ ; \ F = W\sin\alpha$$

Resolving perpendicular to the plane

$$\nwarrow \quad R - W\cos\alpha = 0 \ ; \ R = W\cos\alpha$$

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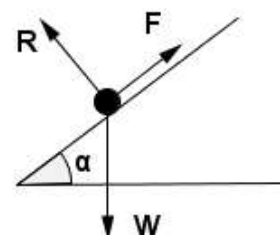
For equilibrium $\quad \dfrac{F}{R} \leq \mu$

$$\dfrac{W \sin \alpha}{W \cos \alpha} \leq \tan \lambda$$

$$\tan \alpha \leq \tan \lambda$$

$$\alpha \leq \lambda$$

7.3 Worked examples

Example 1

A body of weight 9 N which is placed on a rough horizontal plane is pulled by a string inclined at an angle 30° to the horizontal. If it just begins to move when the tension in the string is 6 N, find the coefficient of friction between the body and the plane.

Resolving horizontally

$$\rightarrow \quad 6 \cos 30 - F = 0 \quad ; \quad F = 3\sqrt{3}$$

Resolving vertically

$$\uparrow \quad R + 6 \sin 30° - 9 = 0$$

$$R = 6$$

For limiting equilibrium

$$\mu = \dfrac{F}{R}$$

$$= \dfrac{3\sqrt{3}}{6} = \dfrac{\sqrt{3}}{2}$$
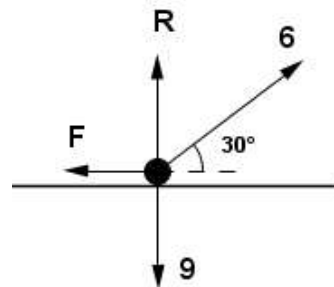
Example 2

A body is placed on a plane of inclination 45° to the horizontal. The coefficient of friction between the body

and the plane is $\dfrac{1}{3}$. A horizontal force 6 N is necessary to prevent the body from sliding down the plane.

(a)   Find the weight of the body.

(b)   If the motion of the body up the plane starts when the force is increased gradually find the value of the force.

(a) $\mu = \dfrac{1}{3}$

Resolving parallel to the plane

$$\nearrow \quad F + 6 \cos 45° - W \sin 45° = 0 \quad ; \quad F = \dfrac{W - 6}{\sqrt{2}}$$

Resolving perpendicular to the plane

$$\nwarrow R - 6 \sin 45° - W \cos 45° = 0 \quad ; \quad R = \dfrac{W + 6}{\sqrt{2}}$$
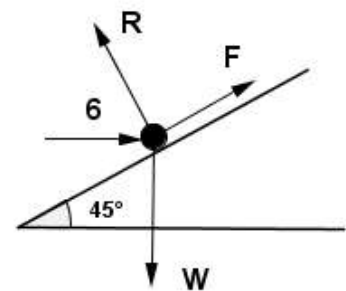
For limiting equilibrium

$$\frac{F}{R} = \mu \quad ; \quad \frac{\dfrac{W-6}{\sqrt{2}}}{\dfrac{W+6}{\sqrt{2}}} = \frac{1}{3}$$

$$\frac{W-6}{W+6} = \frac{1}{3} \quad ; \quad W = 12\,N$$

(b)

Resolving parallel to the plane

$$\swarrow \quad F - P\cos 45° + 12\sin 45° = 0 \quad ; \quad F = \frac{P-12}{\sqrt{2}}$$

Resolving perpendicular to the plane

$$\nwarrow \quad R - P\sin 45° - 12\cos 45° = 0 \quad ; \quad R = \frac{P+12}{\sqrt{2}}$$
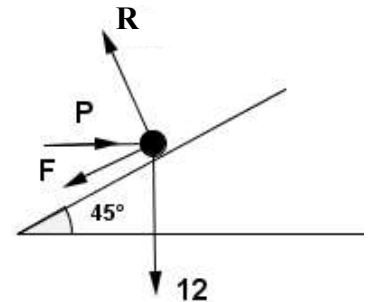
In limiting equilibrium

$$\frac{F}{R} = \mu$$

$$\frac{\dfrac{P-12}{\sqrt{2}}}{\dfrac{P+12}{\sqrt{2}}} = \frac{1}{3}$$

$$\frac{P-12}{P+12} = \frac{1}{3} \quad ; \quad P = 24\,N$$

Equilibrium of a particle on a rough plane

•   **The minimum force required to move a particle on a rough horizontal plane**

Let the weight of the particle be W and the angle of friction λ.

Forces acting on the particle :
  (a)   Weight W
  (b)   Frictional force F
  (c)   Normal reaction R
  (d)   Required force P at an angle θ with the horizontal
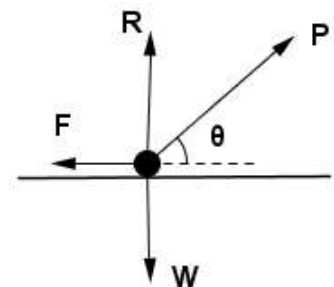
For equilibrium of the particle

Resolving horizontally

$$\rightarrow \quad P\cos\theta - F = 0 \quad ; \quad F = P\cos\theta$$

Resolving vertically

$$\uparrow \quad R + P\sin\theta - W = 0 \quad ; \quad R = W - P\sin\theta$$

For limiting equilibrium

$$\frac{F}{R} = \mu = \tan \lambda$$

$$\frac{P \cos \theta}{W - P \sin \theta} = \frac{\sin \lambda}{\cos \lambda}$$

$$P (\cos \theta \cos \lambda + \sin \theta \sin \lambda) = W \sin \lambda$$

$$P \cos (\theta-\lambda) = W \sin \lambda$$

$$P = \frac{W \sin \lambda}{\cos (\theta-\lambda)}$$

P to be minimum $\cos (\theta - \lambda) = 1$.   This means $\theta = \lambda$

$$\theta = \lambda \text{ and } P_{min} = W \sin \lambda$$

• **When the inclination of the plane is less than the angle of friction, the least force required to move the particle down the plane**

Let α be the inclination of the plane. Since $\alpha < \lambda$, the particle will be in equilibrium.

Let the force applied be P at an angle θ with the plane

For equilibrium of the particle,

Resolving parallel to the plane,

$\nearrow$ $P \cos \theta + W \sin \alpha - F = 0$

Resolving perpendicular to the plane,

$\nwarrow R - W \cos \alpha + P \sin \theta = 0$

At limiting equilibrium

$$\frac{F}{R} = \mu = \tan \lambda$$

$$\frac{P \cos \theta + W \sin \alpha}{W \cos \alpha - P \sin \theta} = \frac{\sin \lambda}{\cos \lambda}$$

$$P(\cos \theta \cos \lambda + \sin \theta \sin \lambda) = W(\sin \lambda \cos \alpha - \cos \lambda \sin \alpha)$$

$$P \cos (\theta-\lambda) = W \sin (\lambda- \alpha)$$

$$P = \frac{W \sin (\lambda-\alpha)}{\cos (\theta-\lambda)}$$

P to be minimum $\cos (\theta-\lambda) = 1$;

i.e. $\theta = \lambda$ and the least value of $P = W \sin (\lambda-\alpha)$

- **When the inclination of the plane is less than the angle of friction, the least force required to move the particle up the plane**

Let the inclination be α. Since $\alpha < \lambda$, the particle will be in equilibrium.

Let the force applied be P at an angle θ with the plane.

For equilibrium,

Resolving parallel to the plane,

$\nearrow$ $P \cos \theta - F - W \sin \alpha = 0$

Resolving perpendicular to the plane,

$\nwarrow$ $R + P \sin \theta - W \cos \alpha = 0$

At limiting equilibrium

$$\frac{F}{R} = \mu = \tan \lambda$$

$$\frac{P \cos \theta - W \sin \alpha}{W \cos \alpha - P \sin \theta} = \frac{\sin \lambda}{\cos \lambda}$$

$$P (\cos \theta \cos \lambda + \sin \theta \sin \lambda) = W (\sin \alpha \cos \lambda + \cos \alpha \sin \lambda)$$

$$P \cos (\theta - \lambda) = W \sin (\alpha + \lambda)$$

$$P = \frac{W \sin (\alpha + \lambda)}{\cos (\theta - \lambda)}$$

P to be minimum $\cos (\theta - \lambda) = 1$;

i.e. $\theta = \lambda$ and the least value of $P = W \sin (\alpha + \lambda)$

- **When the inclination of the plane is greater than the angle of friction, the least force required to move the particle upwards on the plane**

Since $\alpha > \lambda$, the particle will slide down on the plane.

For equilibrium,

Resolving parallel to the plane,

$\nearrow$ $P \cos \theta - F - W \sin \alpha = 0$

Resolving perpendicular to the plane,

$\nwarrow$ $R + P \sin \theta - W \cos \alpha = 0$

At limiting equilibrium

$$\frac{F}{R} = \mu = \tan \lambda$$

$$\frac{P \cos \theta - W \sin \alpha}{W \cos \alpha - P \sin \theta} = \frac{\sin \lambda}{\cos \lambda}$$
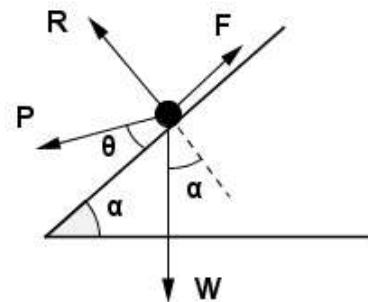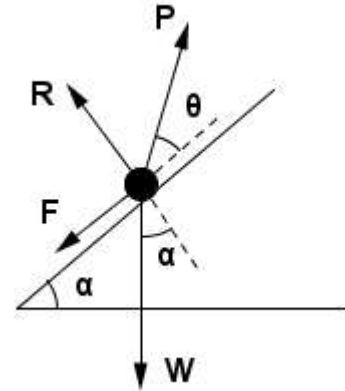
$$P (\cos \theta \cos \lambda + \sin \theta \sin \lambda) = W (\sin \alpha \cos \lambda + \cos \alpha \sin \lambda)$$

$$P \cos (\theta - \lambda) = W \sin (\alpha + \lambda)$$

$$P = \frac{W \sin (\alpha + \lambda)}{\cos (\theta - \lambda)}$$

P to be minimum $\cos (\theta - \lambda) = 1$;

i.e. $\theta = \lambda$ and the least value of $P = W \sin (\alpha + \lambda)$

- **When the inclination of the plane is greater than the angle of friction, the least force required to support the particle**

Let α be the inclination of the plane to the horizontal. Since $\alpha > \lambda$, the particle will slide down on the plane. We have to find the least force to support.

The particle is on the point of moving down the plane. Therefore the frictional force F acts up the plane,

For equilibrium of the particle,

Resolving parallel to the plane,

$\nearrow$ F + P cos θ - W sin α = 0

Resolving perpendicular to the plane,

$\nwarrow$ R + P sin θ - W cos α = 0

At limiting equilibrium, $\quad \dfrac{F}{R} = m = \tan \lambda$

$$\dfrac{W \sin \alpha - P \cos \theta}{P \cos \alpha - W \sin \theta} = \dfrac{\sin \lambda}{\cos \lambda}$$

W (sin α cos λ - cos α sin λ) = P (cos θ cos λ - sin θ sin λ)

P cos ($\theta + \lambda$) = W sin ($\alpha - \lambda$)

$$P = \dfrac{W \sin (\alpha - \lambda)}{\cos (\theta + \lambda)}$$

P to be minimum cos ($\theta + \lambda$) = 1 ;

i.e. $\theta = - \lambda$ and the least value of P = W sin ($\alpha - \lambda$)

$\theta = - \lambda$ means P acts along LM and the least value of P is W sin ($\alpha - \lambda$)
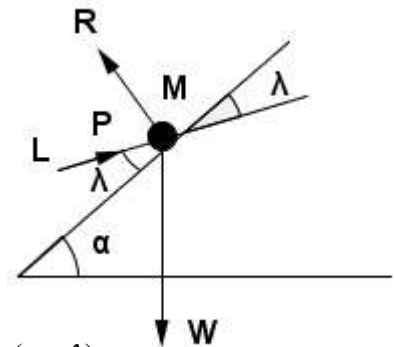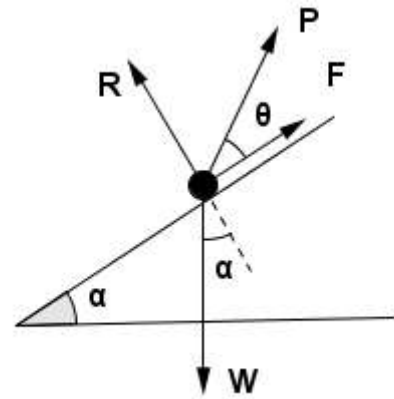
Equilibrium of rigid bodies on rough planes

Example 3

A uniform rod of length 2a and weight W rests one end against a smooth wall and the other end on a rough horizontal floor, the coefficient of friction being μ. If the rod is on the point of slipping show that inclination of the rod to the horizontal is $\tan^{-1}\left(\dfrac{1}{2}\cot \lambda\right)$ and find the reaction at the wall and on the ground, where λ is the angle of friction.

Method I

Let θ be the angle the rod makes with the horizontal.

For equilibrium of the rod AB,

Resolving horizontally,

$\rightarrow$ F - S = 0　　;　　　F = S　　------- ①

Resolving vertically,

$\uparrow$ R - W = 0　　;　　　R = W　　-------- ②

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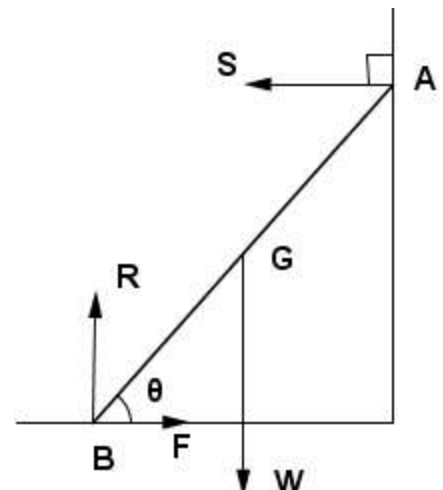
Taking moments about B

$$\text{B} = 0, \qquad S.2a \sin\theta - Wa\cos\theta = 0$$

$$S = \frac{W}{2}\cot\theta \quad \text{--------- ③}$$

$$\text{From ① and ③,} \ F = S = \frac{W}{2}\cot\theta$$

At limiting equilibrium

$$\frac{F}{R} = \mu$$

$$\frac{W\cot\theta}{2} \times \frac{1}{W} = \tan\lambda$$

$$\cot\theta = 2\tan\lambda$$

$$\tan\theta = \frac{1}{2}\cot\lambda$$

$$\theta = \tan^{-1}\left(\frac{1}{2}\cot\lambda\right)$$

$$S = \frac{W}{2}.2\tan\lambda$$

$$= W\tan\lambda$$

Method II

The reaction S at A and weight W of the rod meet at O.

For equilibrium of the rod AB, the resultant R′, of F and R passes through O.

Since the rod is at limiting equilibrium, the angle between R and R′ is λ (angle of friction)

Applying cot rule for triangle AOB

$$\left(BG + GA\right)\cot\left(90° - \theta\right) = BG\cot\lambda - GA\cot 90°$$

$$(a + a)\tan\theta = a\cot\lambda$$

$$2\tan\theta = \cot\lambda$$

$$\tan\theta = \frac{1}{2}\cot\lambda$$

$$\theta = \tan^{-1}\left(\frac{1}{2}\cot\lambda\right)$$

Reaction at the wall is $\ S = F = \dfrac{W}{2}\cot\theta \quad$ (from ③)

$$= W\tan\lambda$$

Reaction at the ground is $= \sqrt{F^2 + R^2}$

$$= \sqrt{\left(W\tan\lambda\right)^2 + W^2}$$

$$= \sqrt{W^2\left(1 + \tan^2\lambda\right)}$$

$$= W\sec\lambda$$
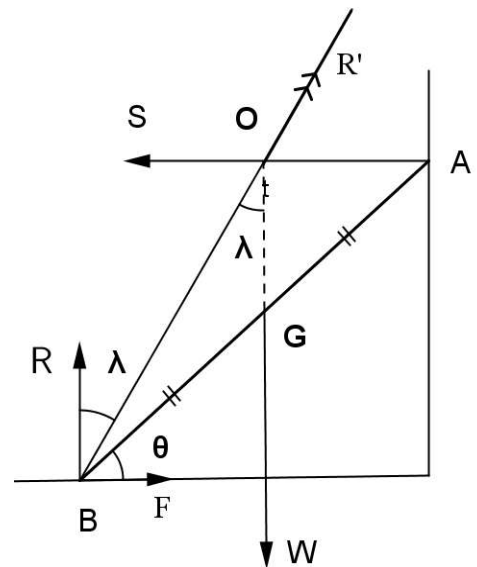
Example 4

A uniform rod rests with one end on a rough ground and the other end on a rough wall. The vertical plane containing the rod is perpendicular to the wall. The coefficient of friction at the wall is μ_1 and ground is μ_2. If the rod is on the point of slipping at both ends, show that the angle the rod makes with horizontal is $\tan^{-1}\left(\dfrac{1-\mu_1\mu_2}{2\mu_2}\right)$.

(i)   The resultant of F_1 and R_1 is S_1.

(ii) The resultant of F_2 and R_2 is S_2.

(iii) Weight of the rod is W

For equilibrium of the rod the three forces S_1, S_2 and W meet at a point O.

Let $\mu_1 = \tan\lambda_1$ and $\mu_2 = \tan\lambda_2$

The angle between R_1 and S_1 is λ_1

The angle between R_2 and S_2 is λ_2

Applying Cot Rule for triangle AOB

$$(AG + GB)\cot(90° - \alpha) = AG\cot\lambda_2 - GB\cot(90° - \lambda_1)$$

$$(1+1)\tan\alpha = \frac{1}{\tan\lambda_2} - \tan\lambda_1$$

$$2\tan\alpha = \frac{1 - \tan\lambda_1\,\tan\lambda_2}{\tan\lambda_2}$$

$$\tan\alpha = \frac{1 - \tan\lambda_1\,\tan\lambda_2}{2\tan\lambda_2}$$

$$\tan\alpha = \left(\frac{1 - \mu_1\mu_2}{2\mu_2}\right)$$

$$\alpha = \tan^{-1}\left(\frac{1 - \mu_1\mu_2}{2\mu_2}\right)$$
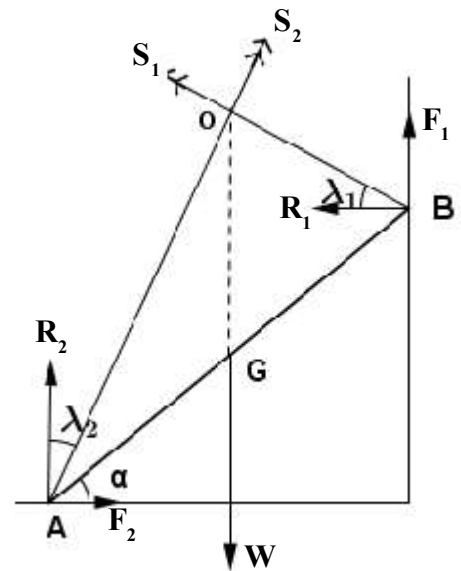
Example 5

A uniform rod AB of weight W and length $2a$ is kept in equilibrium with the end A in contact with a rough vertical wall supported by a light inextensible string of equal length 2a connecting the other end B to a point C on the wall vertically above A. The rod is inclined at an angle θ to the upward vertical and lies in a vertical plane perpendicular to the wall.

Find the tension in the string and show that $\theta \geq \cot^{-1}\left(\dfrac{\mu}{3}\right)$, where μ is the coefficient of friction.

The tension T in the string at B and the weight of the rod W meet at O.

Therefore, for equilibrium of the rod the resultant R^1 of F and R at A passes through O.

$$C\hat{A}B = \theta, \text{ since } BA = BC, \quad B\hat{A}C = B\hat{C}A = \theta$$

$$\therefore \quad A\hat{B}C = 180 - 2\theta$$

For equilibrium of AB,

$$Am \quad = 0$$

$$T.\, AB \sin(180° - 2\theta) - W.AG \sin \theta = 0$$

$$T.2a \sin 2\theta = W.a \sin \theta$$

$$T = \frac{W}{4 \cos \theta}$$

$$= \frac{W \sec \theta}{4}$$

For equilibrium of AB,

Resolving horizontally,

$$\rightarrow R - T \cos(90° - \theta) = 0$$

$$R = T \sin \theta = \frac{W \tan \theta}{4}$$

Resolving vertically,

$$\uparrow \quad T \cos \theta + F - W = 0$$

$$F = W - T \cos \theta$$

$$= W - \frac{W}{4} = \frac{3W}{4}$$

For equilibrium,

$$\frac{F}{R} \leq \mu$$

$$\frac{3W}{4} \times \frac{4}{W \tan \theta} \leq \mu$$

$$3 \cot \theta \leq \mu$$

$$\cot \theta \leq \frac{\mu}{3}$$

$$\theta \geq \cot^{-1}\left(\frac{\mu}{3}\right)$$
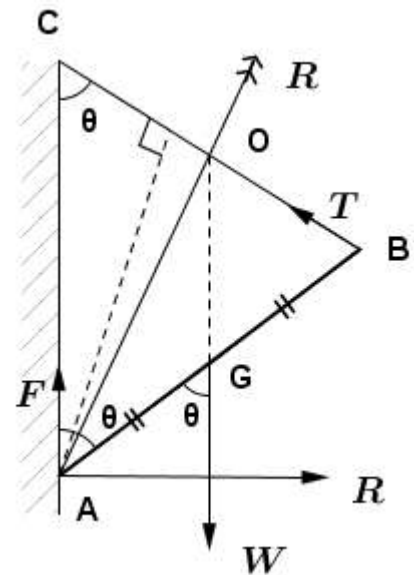
Example 6

A ladder whose centre of gravity is at a distance b from the foot, stands on a rough horizontal ground and leans in equilibrium against a rough cylindrical pipe of radius r fixed on the ground. The ladder projects beyond the point of contact with the pipe and is perpendicular to the axis of the pipe. Let λ be the angle of friction at both points where friction acts, and 2α (such that $b < \cot \alpha$), be the inclination of the ladder to the horizontal. A load of weight equal to that of the ladder is suspended from a point at a distance x measured along the ladder from its foot. The ladder is in limiting equilibrium at both points where friction acts. Show that $(b + x) \sin^2 \alpha \cos 2\alpha = r \sin \alpha \cos \alpha$.

The resultant S_1 of the forces F_1 and R_1 at C,

The resultant S_2 of the forces F_2 and R_2 at A and the resulta weight of the ladder W and the weight W at meet at O.

Since equilibrium is limiting

    (i)   the angle between R_1 and S_1 is λ

    (ii) the angle between R_2 and S_2 is λ

$$AM = b, \quad AC = r \cot \alpha$$

$$AL = x \quad\quad AM = b,$$

Therefore $AG = AL + LG = x + \dfrac{b-x}{2} = \dfrac{b+x}{2}$

Now $AG = \dfrac{b+x}{2}$ and $GC = r \cot \alpha - \left(\dfrac{b+x}{2}\right)$

Appling Cot Rule for the triangle ACO,

$$\left(AG + GC\right) \cot \left(90° - 2\alpha\right) = GC \cot \left[90° - \left(\lambda + 2\alpha\right)\right] - AG \cot \left(90° + \lambda\right)$$

$$AC \tan 2\alpha = GC \tan \left(\lambda + 2\alpha\right) + AG \tan \lambda$$

$$r \cot \alpha . \tan 2\alpha = \left[r \cot \alpha - \left(\dfrac{b+x}{2}\right)\right] \tan \left(\lambda + 2\alpha\right) + \left(\dfrac{b+x}{2}\right) \tan \lambda$$

$$r \cot \alpha \left[\tan 2\alpha - \tan \left(\lambda + 2\alpha\right)\right] = \left(\dfrac{b+x}{2}\right) \left[\tan \lambda - \tan \left(\lambda + 2\alpha\right)\right]$$

$$r \cot \alpha \left[\dfrac{\sin 2\alpha}{\cos 2\alpha} - \dfrac{\sin \left(\lambda + 2\alpha\right)}{\cos \left(\lambda + 2\alpha\right)}\right] = \left(\dfrac{b+x}{2}\right)\left[\dfrac{\sin \lambda}{\cos \lambda} - \dfrac{\sin \left(\lambda + 2\alpha\right)}{\cos \left(\lambda + 2\alpha\right)}\right]$$

$$r \dfrac{\cos \alpha}{\sin \alpha}\left[\dfrac{\sin \left[2\alpha - \left(\lambda + 2\alpha\right)\right]}{\cos 2\alpha . \cos \left(\lambda + 2\alpha\right)}\right] = \left(\dfrac{b+x}{2}\right)\left[\dfrac{\sin \left[\lambda - \left(\lambda + 2\alpha\right)\right]}{\cos \lambda . \cos \left(\lambda + 2\alpha\right)}\right]$$

$$r \dfrac{\cos \alpha}{\sin \alpha} \times \dfrac{\sin \left(-\lambda\right)}{\cos 2\alpha} = \left(\dfrac{b+x}{2}\right)\dfrac{\sin \left(-2\alpha\right)}{\cos \lambda}$$

$$\dfrac{r \cos \alpha . \sin \lambda}{\sin \alpha . \cos 2\alpha} = \left(\dfrac{b+x}{2}\right)\dfrac{2 \sin \alpha \cos \alpha}{\cos \lambda}$$

$$r \sin \lambda \cos \lambda = \left(b + x\right) \sin^2 \alpha . \cos 2\alpha$$
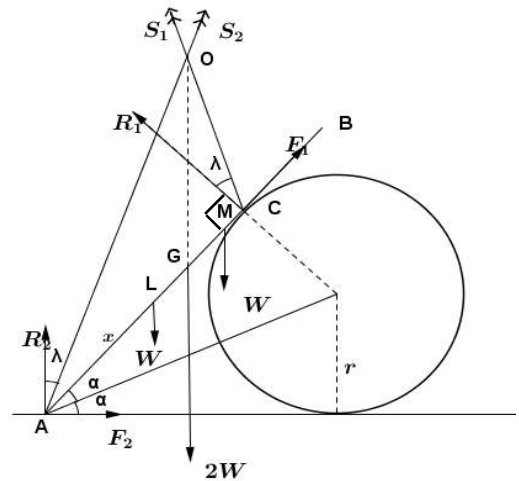
Example 7

A particle A of weight w, resting on a rough horizontal floor is attached to one end of a light inextensible string wound round a right circular cylinder of radius a and weight W, that rests on the floor, touching it along a generator through a point B. The other end of the string is fastened to the cylinder. The vertical plane through the string is perpendicular to the axis of the cylinder, passes through the centre of gravity of the cylinder and intersects the floor along AB, as shown in the figure.
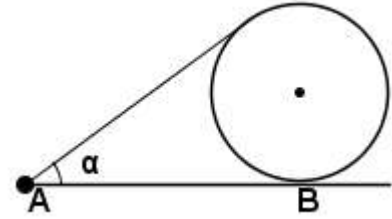
The string is just taut and makes an angle α with AB. The floor is rough enough to prevent the cylinder from moving at B. A couple of moment G is applied to the cylinder so that the particle is in limiting equilibrium. If μ is the coefficient of friction between the particle and floor, show that the tension in the

string is $\dfrac{\mu W}{(\cos \alpha + \mu \sin \alpha)}$.

By taking moments about B, find the value of G.

For equilibrium of the system,

Resolving horizontally

$$\rightarrow F_2 - F_1 = 0$$
$$F_2 = F_1$$

Resolving vertically

$$\uparrow \quad R_1 + R_2 - W - w = 0$$
$$R_1 + R_2 = W + w$$
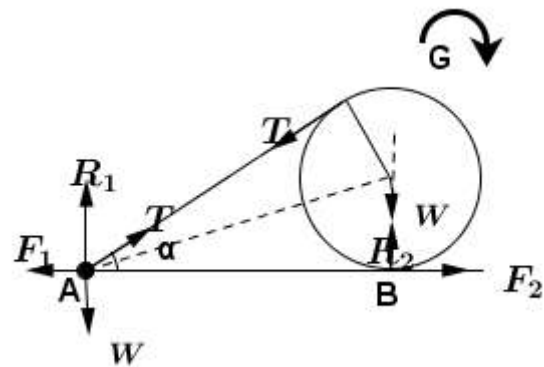
For equilibrium of the particle,

Resolving horizontally,

$$\rightarrow T \cos \alpha - F_1 = 0 \quad ; \quad F_1 = T \cos \alpha$$

Resolving vertically,

$$\uparrow \quad R_1 + T \sin \alpha - w = 0 \quad ; \quad R_1 = w - T \sin \alpha$$

At limiting equilibrium,

$$\frac{F_1}{R_1} = \mu$$

$$\frac{T \cos \alpha}{w - T \sin \alpha} = \mu \quad ; \quad T (\cos \alpha + \mu \sin \alpha) = \mu w$$

$$T = \frac{\mu w}{\cos \alpha + \mu \sin \alpha}$$

For equilibrium of the cylinder,

$$B m \qquad T(a + a \cos \alpha) - G = 0$$

$$G = T . a (1 + \cos \alpha)$$

$$= \frac{\mu w a (1 + \cos \alpha)}{\cos \alpha + \mu \sin \alpha}$$

Example 8

A uniform rod of length a and weight W rests in a vertical plane inside a fixed rough hemispherical bowl of radius a. The rod is in limiting equilibrium inclined at an angle θ to the horizontal, and the coefficient of friction is $\mu\left(<\sqrt{3}\right)$. Show that the reaction at the lower end of the rod is $\dfrac{W\cos\theta}{\sqrt{3}-\mu}$ and find the reaction

at the upper end. Hence show that $\tan\theta = \dfrac{4\mu}{3-\mu^2}$.

Since ther rod is in limiting equilibrium,

$$F_1 = \mu R \quad \text{and} \quad F_2 = \mu S$$

For equilibrium of AB, Taking moment about B

BM $-R.\,a\sin 60° + \mu R .\, a\sin 30° + w.\dfrac{a}{2}\cos\theta = 0$

$$- R\frac{\sqrt{3}}{2} + \mu R.\frac{1}{2} + w.\frac{1}{2}\cos\theta = 0$$

$$R\left(\sqrt{3} - \mu\right) = w\cos\theta \quad ; \quad R = \frac{w\cos\theta}{\sqrt{3}-\mu} \text{---------(1)}$$
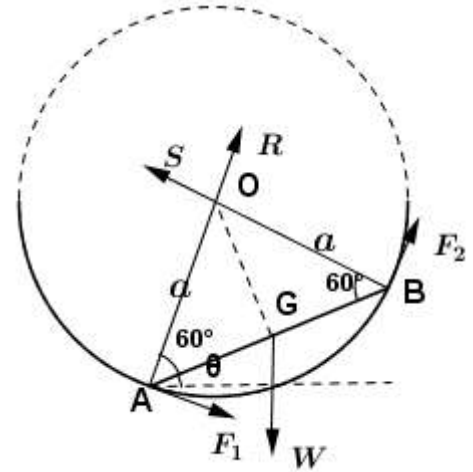
Taking moment about A

AM $S.a\sin 60° + \mu S.a\sin 30° - w.\dfrac{a}{2}\cos\theta = 0$

$$\frac{\sqrt{3}S}{2} + \frac{\mu S}{2} = \frac{w\cos\theta}{2}$$

$$S = \frac{w\cos\theta}{\left(\sqrt{3}+\mu\right)} \text{--------------(2)}$$

Taking moment about O

OM $F_1.a + F_2.a - w\left(\dfrac{a}{2}\cos\theta - a\cos(60+\theta)\right) = 0$

$$\mu(R+S) = w\left(\frac{1}{2}\cos\theta - \frac{1}{2}\cos\theta + \frac{\sqrt{3}}{2}\sin\theta\right)$$

$$\mu\left[\frac{w\cos\theta}{\sqrt{3}-\mu} + \frac{w\cos\theta}{\sqrt{3}+\mu}\right] = \frac{\sqrt{3}}{2}w\sin\theta$$

$$\frac{\mu\cos\theta \times 2\sqrt{3}}{3-\mu^2} = \frac{\sqrt{3}}{2}w\sin\theta$$

$$\tan\theta = \frac{4\mu}{3-\mu^2}$$

Example 9

A uniform solid hemisphere of weight W is placed with its curved surface on a rough plane inclined at an angle α to the horizontal. When a small weight w is attached to a point on the circumference of its plane surface, the plane surface becomes horizontal. Show that if μ is the coefficient of friction, then

$$\mu = \frac{w}{\sqrt{W(W+2w)}} = \tan \alpha .$$

Centre of gravity of the hemisphere is at G and $OG = \frac{3}{8}a$.

The forces F and R act at C on the hemisphere.

The resultant of W and w also should pass through C.

Taking moments about N,

$$W . ON - w . BN = 0$$

$$W . ON = w (a - ON)$$

$$(W - w) . ON = w . a$$

$$ON = \frac{w . a}{W + w}$$
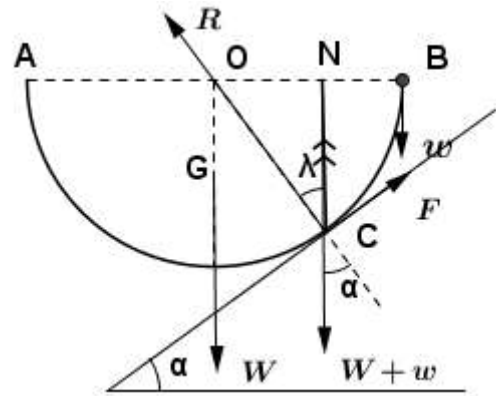
For equilibrium of the system the resultant of F and R must be equal to $(W+w)$ in magnitude and opposite in direction.

Since the equilibrium is limiting, $O\hat{C}N = \lambda$

$$\tan \lambda = \frac{ON}{CN} = \frac{ON}{\sqrt{a^2 - ON^2}}$$

$$= \frac{\dfrac{w . a}{W + w}}{\sqrt{a^2 - \dfrac{w^2 a^2}{(W+w)^2}}}$$

$$= \frac{w}{\sqrt{W^2 + 2Ww}}$$

$$\mu = \frac{w}{\sqrt{W(W+2w)}}$$

$$= \tan \alpha \ (\text{since } \lambda = \alpha)$$

Example 10

Two uniform rods AB and BC of equal length and of weights W and w ($W > w$) respectively are freely jointed at B. The rods rest in equilibrium in a vertical plane with $A\hat{B}C = \dfrac{\pi}{2}$ and the ends A and C on a rough horizontal ground. If μ is the coefficient of friction between the rods and the ground, show that the least possible value of μ is $\dfrac{W+w}{W+3w}$ in order to preserve the equilibrium. If $\mu = \dfrac{W+w}{W+3w}$, prove that the slipping is about to occur at C but not at A.

For equilibrium of the system,

$\rightarrow \quad F_1 - F_2 = 0 \quad ; \quad F_1 = F_2 \ (= F, \text{ say})$

$\uparrow \quad R + S - W - w = 0$

$\qquad R + S = W + w \qquad \text{-----------------(1)}$

$A_m = 0$

S. $4a \cos 45° - w. 3a \cos 45° - Wa \cos 45° = 0$

$S = \dfrac{W+3w}{4} \quad$ and $\quad R = \dfrac{3W+w}{4}$

For equilibrium of AB, $B_m = 0$

$F_1.2a \sin 45° + Wa \cos 45° - R.2a \cos 45° = 0$

$\qquad\qquad 2F_1 + W - 2R = 0$

$$F_1 = R - \dfrac{W}{2}$$

$$= \dfrac{3W+w}{4} - \dfrac{W}{2}$$

$$= \dfrac{W+w}{4}$$

$$F_1 = F_2 = F = \dfrac{W+w}{4}$$
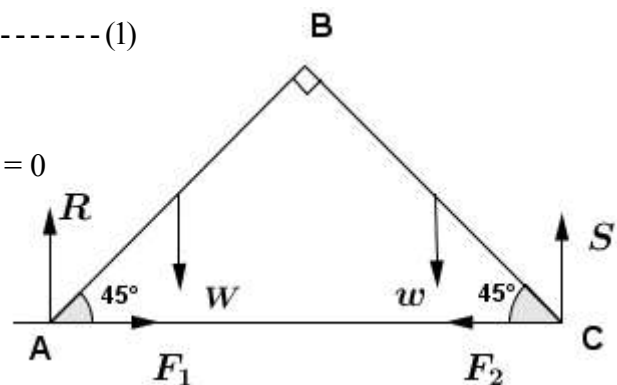
For equilibrium of the system

$$\dfrac{F_1}{R} \leq \mu \ , \quad \dfrac{F_2}{S} \leq \mu$$

$$\dfrac{F_1}{R} = \dfrac{\frac{W+w}{4}}{\frac{3W+w}{4}} = \dfrac{W+w}{3W+w} \leq \mu$$

$$\dfrac{F_2}{S} = \dfrac{\frac{W+w}{4}}{\frac{W+3w}{4}} = \dfrac{W+w}{W+3w} \leq \mu$$

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Now, $R - S = \dfrac{3W + w}{4} - \dfrac{W + 3w}{4} = \dfrac{W - w}{2} > 0$

i.e. $R > S$

$$R > S \ (> 0) \ \Rightarrow \ \dfrac{1}{R} < \dfrac{1}{S}$$

$$\Rightarrow \ \dfrac{F}{R} < \dfrac{F}{S}$$

$$\Rightarrow \ \dfrac{F_1}{R} < \dfrac{F_2}{S}$$
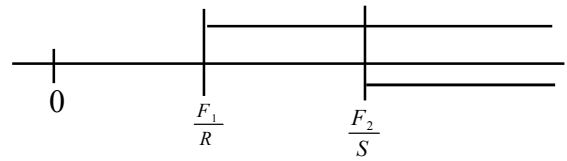
For equilibrium to be possible, $\dfrac{F_1}{R} \leq \mu$, $\dfrac{F_2}{S} \leq \mu$

The least possible value is $\dfrac{F_2}{S} = \dfrac{W + w}{W + 3w}$

If $\mu = \dfrac{W + w}{W + 3w}$, slipping first occurs at C

Example 11

A uniform plank AB of length 4ℓ and weight W rests with one end A on level ground and leans against a cylinder of radius ℓ such that the point of contact between the plank and the cylinder is at a distance 3ℓ from A. The cylinder is uniform and of weight W and rests on the ground with its axis perpendicular to the vertical plane containing the plank. Find the frictional force at each point of contact and show that for

equilibrium to be possible $\mu \geq \dfrac{8}{21}$, where μ is the coefficient of friction.

For equilibrium of the system,

Resolving horizontally

$\rightarrow F_1 - F_2 = 0$; $F_1 = F_2$

Resolving vertically,

$\uparrow R_1 + R_2 - 2W = 0$

$R_1 + R_2 = 2W$ .... ①
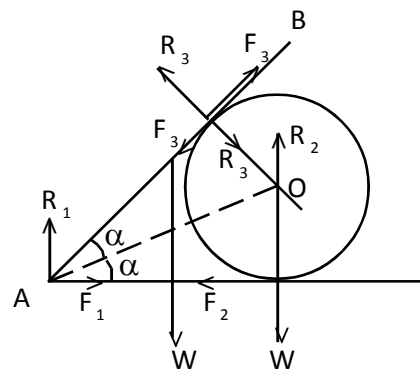
For equilibrium of the sphere,

Om $F_2 . a - F_3 . a = 0$; $F_2 = F_3$

Hence $F_1 = F_2 = F_3$ .... ②

For equilibrium of the rod AB,

Am $R_3 . 3\ell - W . 2\ell \cos 2\alpha = 0$

$R_3 = \dfrac{2W \cos 2\alpha}{3} = \dfrac{8W}{15}$ .... ③

For equilibrium of the system,

Am $R_2 . 3\ell - W . 3\ell - W . 2\ell \cos 2\alpha = 0$

$3R_2 = 3W + 2W \times \dfrac{4}{5}$

$R_2 = \dfrac{23W}{15}$; From ① $R_1 = \dfrac{7W}{15}$ .... ④

For equilibrium of AB,

Resolving along AB,

$$\nearrow F_3 + F_1 \cos 2\alpha + R_1 \sin 2\alpha - W \sin 2\alpha = 0$$

$$F_3 + F_1 \cos 2\alpha = \left(W - \frac{7W}{15} \right) \sin 2\alpha$$

$$F_1 (1 + \cos 2\alpha) = \frac{8W}{15} \times \frac{3}{5} = \frac{24W}{75} \quad (\text{since } F_1 = F_3)$$

$$F_1 \left(1 + \frac{4}{5} \right) = \frac{24W}{75}; \quad F_1 = \frac{8W}{45}$$

For equilibrium to be possible,

$$\frac{F_1}{R_1} \leq \mu; \quad \frac{F_2}{R_2} \leq \mu, \frac{F_3}{R_3} \leq \mu$$

$$\frac{8W}{45} \times \frac{15}{7W} \leq \mu; \quad \frac{8W}{45} \times \frac{15}{23W} \leq \mu; \quad \frac{8W}{45} \times \frac{15}{8W} \leq \mu$$

i.e $\quad \mu \geq \frac{8}{21}, \mu \geq \frac{8}{69}; \quad \mu \geq \frac{1}{3}$

Hence for equilibrium to be possible $\mu \geq \frac{8}{21}$
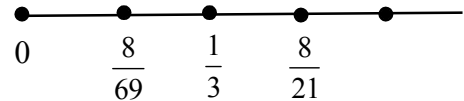
Example 12

An equilateral triangle ABC rests in a vertical plane with the side BC on a rough horizontal plane. A gradually increasing horizontal force is applied on its highest vertex A, in the plane of the triangle. Prove that the triangle will slide before it tilts if the coefficient of friction be less than $\frac{\sqrt{3}}{3}$.

Method I

Forces acting on the triangle ABC are

(i)   Weight W at G

(ii)   Horizontal force P at A

(iii) Frictional force F and normal reaction at A

If the triangle topples, it topples about C.

At the point of toppling the normal reaction acts at C.

Weight *W* at G and the horizontal force P at A meet at A.

Therefore, the resultant S of F and R passes through A. (along CA)

Let θ be the angle between R and S.

(i)   If $\lambda < \theta$, slides before toppling

(ii)   If $\lambda > \theta$, topples before sliding
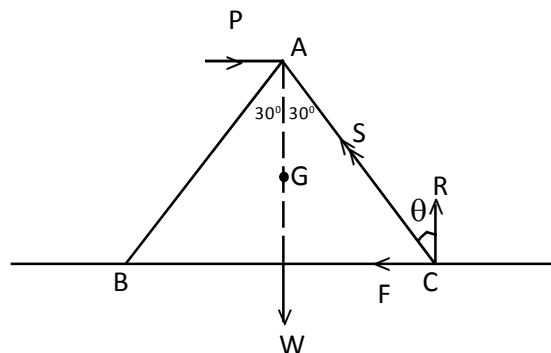
If $\lambda < \theta$, then $\tan \lambda < \tan \theta$

i.e. $\tan \lambda < \tan 30°$

$\tan \lambda < \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$; $\mu < \dfrac{\sqrt{3}}{3}$. Hence if $\mu < \dfrac{\sqrt{3}}{3}$, the triangle will slide before it topples.

Method II

For equilibrium of the triangle ABC.

Resolving horizontally,

$\rightarrow P - F = 0$; $F = P$ .... ①

Resolving vertically

$\uparrow R - W = 0$; $R = W$ .... ②

At limiting equilibrium

$\dfrac{F}{R} = \mu$; $\dfrac{P}{W} = \mu$; $P = \mu W$

For equilibrium of the triangle ABC,

$\rightarrow P - F = 0$; $F = P$

$\uparrow R - W = 0$; $R = W$

At the point of toppling R will act at C

Taking moments about B

Bm $\quad R \times 2a - P\sqrt{3}a - W.a = 0$

$$P = \dfrac{W}{\sqrt{3}}$$

When $P = \mu W$, lamina begins to slide.

When $P = \dfrac{W}{\sqrt{3}}$, lamina toples about C.

If $\mu W < \dfrac{W}{\sqrt{3}}$, lamina will slide before it topples.

ie If $\mu < \dfrac{1}{\sqrt{3}}$, lamina will slide before it topples.
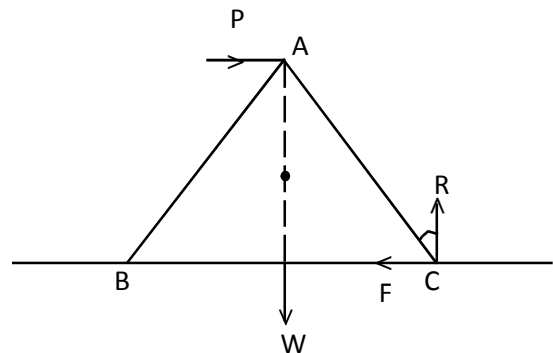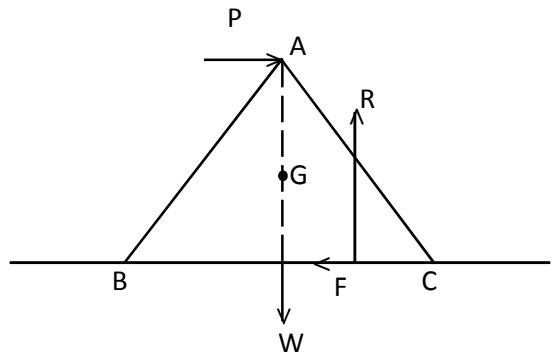
7.4 Exercises

1. Find the least force which will move a mass of 80 kg up a rough plane inclined to the horizontal at 30°. The coefficient of friction is $\dfrac{3}{4}$.

2. If the least force which will move a weight up a plane of inclination α is twice the least force which will just prevent the weight from slipping down the plane, show that the coefficient of friction between the weight and the plane is $\dfrac{1}{3}\tan\alpha$.

3. The least force which will move a weight up an inclined plane is P. Show that the least force, acting parallel to the plane, which will move the weight upwards is $P\sqrt{1+\mu^2}$, where μ is the coefficient of friction.

4. The force P acting along a rough inclined plane is just sufficient to maintain a body on the plane, the angle of friction λ being less than α, the angle of plane. Prove that the least force, acting along the plane, sufficient to drag the body up the plane is $P\dfrac{\sin(\alpha+\lambda)}{\sin(\alpha-\lambda)}$.

5. A uniform ladder rests against a vertical wall at an angle 30° to the vertical. If it is just on the point of slipping down find the coefficient of friction assuming it to be the same for the wall and the ground.

6. A uniform ladder of weight w rests on a rough horizontal ground and against a smooth vertical wall inclined at an angle α to the horizontal. Prove that a man of weight W can climb up the ladder without the ladder slipping, if $\dfrac{w}{W} > \dfrac{2(1 - \mu\tan\alpha)}{2\mu\tan\alpha - 1}$

7. A straight uniform beam of length 2ℓ rests in limiting equilibrium in contact with a rough vertical wall of height h, with one end on a horizontal plane and the other end projecting beyond the wall. If both the wall and the plane are equally rough, prove that λ, the angle of friction is given by $h.\sin 2\lambda = \ell\sin\alpha\cos 2\alpha$ where α is the inclination of the beam to the horizontal.

8. A uniform ladder rests with its ends against a rough vertical wall and an equally rough horizontal ground, the coefficient of friction at both points of contacts is $\dfrac{1}{3}$. Prove that if the inclination of the ladder to the vertical is $\tan^{-1}\dfrac{1}{2}$, a weight equal to that of the ladder cannot be attached to it at a point more than $\dfrac{9}{10}$ th of the distance from the foot of the ladder without destroying the equilibrium.

9. A heavy uniform rod of length $2a$ lies over a rough peg with one extremity leaning against a rough vertical wall. If ℓ be the distance of the peg from the wall and the point of contact of the rod with the wall is above the peg, if the rod is on the point of sliding downwards show that $\sin^3 \theta = \dfrac{c}{a}\cos^2 \lambda$ where λ is the angle of friction at both contact points and θ is the angle between the rod and downward vertical.

10. A uniform ladder of length ℓ rests on a rough horizontal ground with its upper end projecting slightly over smooth horizontal rail at a height a. If the ladder is about to slip and λ is the angle of friction on the ground, prove that $\tan \lambda = \dfrac{a\sqrt{\ell^2 - a^2}}{\ell^2 + a^2}$

11. A uniform rod is in limiting equilibrium, one end resting on a rough horizontal plane and the other on an equally rough plane inclined an angle α to the horizontal, λ be the angle of friction and the rod be in a vertical plane, show that the rod is inclined to be horizontal at an angle $\tan^{-1}\left[\dfrac{\sin(\alpha - 2\lambda)}{2\sin\lambda \sin(\alpha - \lambda)}\right]$

12. A uniform rod is placed within a fixed rough vertical circular loop. If the rod subtends an angle of 60° at the center of the loop and coefficient of friction is $\dfrac{1}{\sqrt{3}}$, show that in the position of limiting equilibrium the inclination of the rod to the horizontal is $\sin^{-1}\sqrt{\dfrac{3}{7}}$.

13. Two equal uniform rods AC, CB are freely joined at C and rests in a vertical plane with the ends A,B in contact with a rough horizontal plane. If the equilibrium is limiting and the coefficient of friction is μ. Show that $\sin A\hat{C}B = \dfrac{4\mu}{1 + \mu^2}$.

14. A uniform lamina in the shape of an equilateral triangle rests with one vertex on a horizontal plane and the other vertex against a smooth vertical wall. The vertical plane containg the lamina is perpendicular to the wall. Show that the least angle that its edge through these vertices can make with the horizontal plane is given by $\cot \theta = 2\mu + \dfrac{1}{\sqrt{3}}$, μ being the coefficient of friction.

15. A uniform ladder AB of length $2a$ and weight W rests with one end A on a rough horizontal floor and the other end B against a rough vertical wall, μ being the coefficient of friction at both ends of the ladder. The ladder is in inclined to the floor at an angle $\dfrac{\pi}{4}$ and a small cat of weight nW gently climb up the ladder, starting from A. Show that, in the position of limiting equilibrium of the ladder, the cat has climbed a distance $\dfrac{a}{n(1+\mu^2)}\left[\mu^2(1+2n) + 2\mu(1+n) - 1\right]$ along the ladder.

Given further that $\mu = \dfrac{1}{2}$, show that the cat reach the top of the ladder before the ladder slips, if

$n < \dfrac{1}{4}$. what happens if $n = \dfrac{1}{4}$

16. A uniform ladder AB of length ℓ and weight w rests with end A on a rough horizontal ground and with the other end B against a smooth vertical wall. The ladder is in a vertical plane perpendicular to the wall and is inclined an angle of to the ground. The coefficient of friction between the ladder and the ground is μ. A force P is applied horizontally towards the wall at the point C on the ladder with $AC = a(<\ell)$ so that limiting equilibrium is attained with the ladder on the point of sliding towards the wall. Show that $\mathrm{P} = \dfrac{\ell w}{2(\ell - a)}(2\mu + \tan\alpha)$

17. Two uniforms rod AB, BC of equal weight but different lengths, are freely jointed together at B and placed in a vertical plane over two equally rough fixed pegs in the same horizontal line. The inclination of the rods to the horizontal are α, β and they are both on the point of slipping. Prove that the inclination θ to the horizontal of the reaction at the hinge is given by $2\tan\theta = \cot(\beta + \lambda) - \cot(\alpha - \lambda)$ where λ is the angle of friction at the pegs.
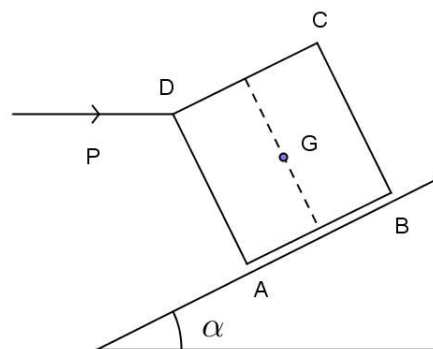
18. Two uniform equal ladders of length ℓ are hinged at the top and rest on a rough floor forming an isosceles triangle with the floor of vertical angle 2θ A man whose weight is n times that of either ladder goes slowly up one of them. Calculate the reaction at the floor when his distance from the top is x, and show that slipping begins when $\dfrac{nx}{\ell} = \dfrac{2\mu - \tan\theta}{\mu - \tan\theta} + n$

19. A smooth cylinder of radius a is fixed on a rough horizontal table with its axis parallel to the table. A uniform rod ACB of length 6a and mass M rests in equilibrium with the end A on the table and the point C touching the cylinder. The vertical plane containing the rod is perpendicular to the axis of the cylinder and the rod makes an angle 2θ with the table.
   a) Show tha the magnitude of the force exerted by the cylinder on the rod is $3Mg\cos 2\theta.\tan\theta$
   b) Show also that μ, the coefficient of friction between the rod and the table, is given by $\mu(\cot\theta - 3\cos^2 2\theta) = 3\sin 2\theta\cos 2\theta$, if the equilibrium is limitimg

20. ABCD represents the central vertical cross section of a uniform cube of side 2a and weight W. The cube is placed on a rough plane of inclination α to the horizontal. A gradually increasing horizontal force P is applied at the point D as shown in the diagram, the coefficient od friction between the cube and the plane being μ. Find the range of value of μ so that the equilibrium is broken by moving up the plane given $\tan\alpha = \dfrac{1}{2}$.
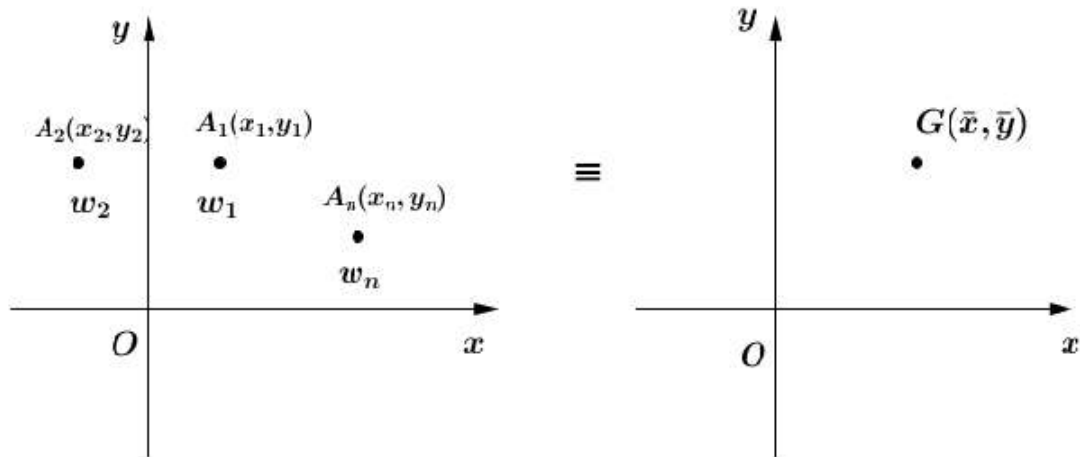
8.0 Centre of Gravity

8.1 Centre of gravity of system of particles

Centre of gravity of a body or a system of particles rigidly connected together is the point through which the line of action of the weight of the body always passes in whatever the position the body is placed.

Centre of gravity of system of particles

Let the particles of weights w_1, w_2 .... w_n be placed at points A_1, A_2, ... A_n lying in one plane. Let the coordinates of these points referred to rectangular axis OX, OY be (x_1, y_1), (x_2, y_2) ....(x_n, y_n)

Let $(\bar{x}, \bar{y})$ be the coordinate of the centre of gravity referred to OXY.

Then weights of the particles form a system of parallel forces, whose resultant $(w_1 + w_2 + ... + w_n)$ acts at $(\bar{x}, \bar{y})$. Suppose the plane to be horizontal, and taking moments about OX and OY for the forces and the resultant, we have

$$\bar{x}\left(w_1 + w_2 + ... + w_n\right) = w_1 x_1 + w_2 x_2 + ... + w_n x_n$$

$$\bar{x} = \frac{\sum\limits_{i=1}^{n} w_i x_i}{\sum\limits_{i=1}^{n} w_i}$$

$$\bar{y}\left(w_1 + w_2 + ... + w_n\right) = w_1 y_1 + w_2 y_2 + ... + w_n y_n$$

$$\bar{y} = \frac{\sum\limits_{i=1}^{n} w_i y_i}{\sum\limits_{i=1}^{n} w_i}$$
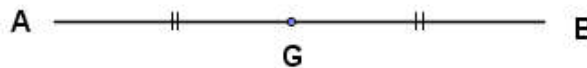
Note :

In uniform bodies centre of gravity, centre of mass, and the centroid are usually the same.

8.2 Centre of gravity of uniform bodies

Centre of gravity of a uniform rod

AB is a uniform rod and G is the midpoint of AB.

Then G is the centre of gravity of the rod AB

Centre of gravity of a uniform triangular lamina

Let AB be a triangular lamina. Suppose it is divided into a very large number of narrow strips, such as PQ parallel to BC.

Centre of gravity of each strip is at its midpoint. Hence the centre of gravity of the whole triangle lies on the line going through the midpoints of the strips.

Thus the centre of gravity is in the median AD

Similarly the centre of gravity lies on the medians through B and C.

Therefore, the centre of gravity of the lamina is the point of intersection

of the medians where $\dfrac{AG}{GD} = \dfrac{2}{1}$
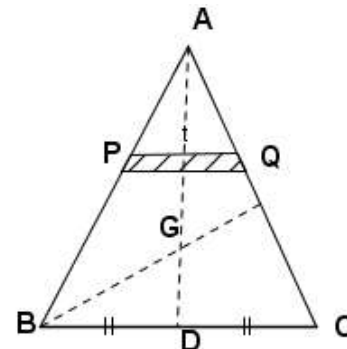
Centre of gravity is the point on the median at a distance equal to two thirds from the vertex.

Centre of gravity of three equal particles placed at the vertices of a triangle
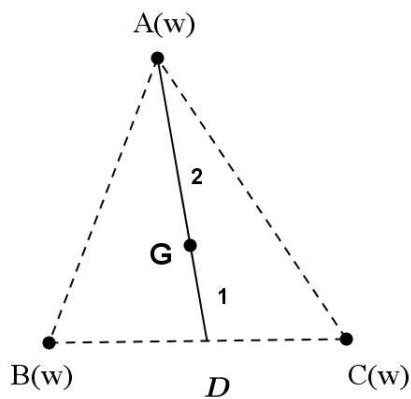
The weights *w* at B and C are equivalent to 2*w* at D where D is the midpoint of BC.

Now the system is equivalent to *w* at A and 2*w* at D.

The weights *w* at A and 2*w* at D are equivalent to 3*w* at G.

Hence centre of gravity of this system is the intersection point of the medians.

The centre of gravity of any uniform triangular lamina is the same as that of three equal particles placed at the vertices of the triangle.

Centre of gravity of a uniform parallelogram lamina

Centre of gravity of a parallelogram lamina is the intersection point of the diagonals.
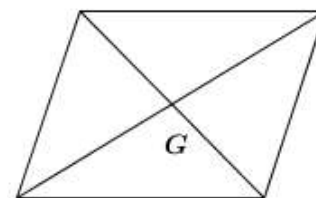
Centre of gravity of a uniform circular ring

The circular ring is symmetric about any diameter. Therefore, the centre of gravity of the circular ring is the point where the diameters meet, i.e. the centre of the circular ring.

8.3 Worked examples

Example 1

One side of a rectangle is twice of the other. On the longer side, an equilateral triangle is described. Find the centre of gravity of the lamina formed by the rectangle and the triangle.

Let AB = a, then AE = $2a$

By symmetry centre of gravity of the lamina lies on MC.

where M is the mid point of AE

Area of ABDE $= 2a^2$

Area of BCD $= \sqrt{3}a^2$

Let w be the weight of unit area.

Lamina	weight	Centre of gravity from M along MC
ABDE	$2a^2w$	$\dfrac{a}{2}$
BDC	$\sqrt{3}a^2w$	$a + \dfrac{1}{3}\sqrt{3}a$
ABCDE	$\left(2+\sqrt{3}\right)a^2w$	$\bar{x}$

Take moment about AE.

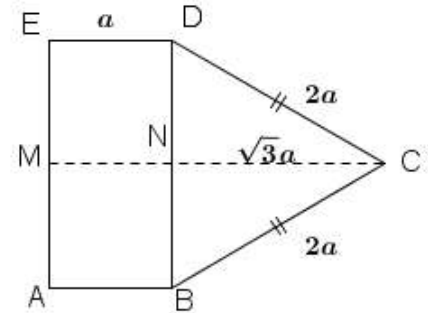$$AE \qquad \left(2+\sqrt{3}\right)a^2w\,\bar{x} = 2aw\,\frac{a}{2} + \sqrt{3}a^2w\left(a + \frac{\sqrt{3}a}{3}\right)$$

$$\left(2+\sqrt{3}\right)\bar{x} = a + \sqrt{3}a + a$$

$$= \left(2+\sqrt{3}\right)a$$

$$\bar{x} = a$$

Hence centre of gravity is N , midpoint of BD

Example 2

The figure shows a uniform lamina ABCDE where ABDE is a rectangle and BCD is a right angled triangle. Find the centre of gravity of the above lamina. If this lamina is suspended from C, find the angle between CE and the vertical.

Area of ABDE = 15 x 12 = 180 cm²

Area of BCD $= \dfrac{1}{2}\times 12\times 6 = 36$ cm²
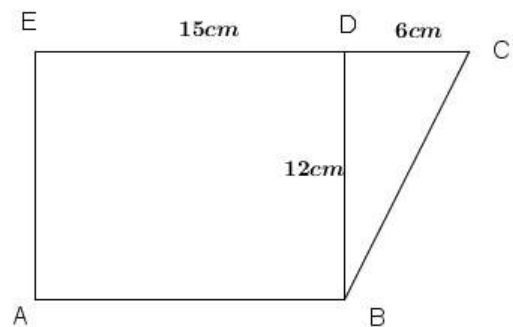
Let w be the weight per unit area

Lamina	Weight	Distance of centre of gravity	
		from AE	from AB
ABDE	$180w$	$\dfrac{15}{2}$ cm	6 cm
BCD	$36w$	$15 + \dfrac{1}{3}\times 6 = 17$ cm	$\dfrac{2}{3}\times 12 = 8$ cm
ABCE	$216w$	$\bar{x}$	$\bar{y}$

Taking moment about AE,

$$216w\,\bar{x} = 180w \times \frac{15}{2} + 36w \times 17$$
$$12\,\bar{x} = 75 + 34$$
$$= 109$$
$$\bar{x} = \frac{109}{12}\ cm$$

Taking moment about AB

$$216w\,\bar{y} = 180w \times 6 + 36w \times 8$$
$$12\,\bar{y} = 60 + 16$$
$$= 76$$
$$\bar{y} = \frac{19}{3}\ cm$$

Centre of gravity is at the distance $\frac{19}{3}$ cm from AE and $\frac{109}{12}$ cm from AB

When the lamina hangs freely from C, CG is vertical.

$$\tan\theta = \frac{MG}{CM}$$
$$= \frac{12 - \bar{y}}{21 - \bar{x}}$$
$$= \frac{12 - \frac{19}{3}}{21 - \frac{109}{12}}$$
$$= \frac{68}{143}$$
$$\theta = \tan^{-1}\left(\frac{68}{143}\right)$$
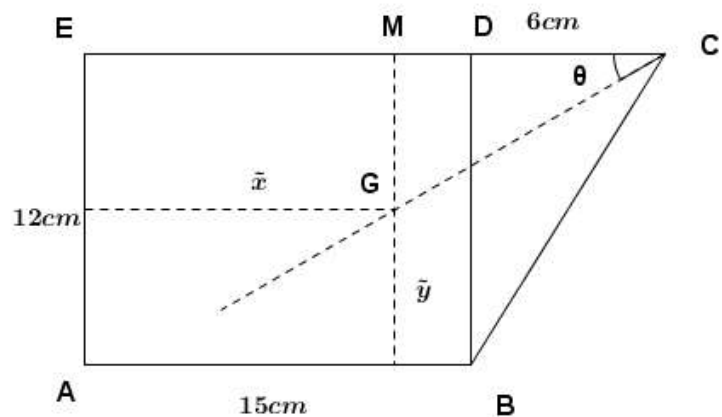
Example 3

Particles of weights 5, 7, 6, 8, 4 and 9 N are placed at the angular points of a regular hexagon taken in order. Show that the centre of gravity coincides with the centre of hexagon.

Let the length of each side of the hexagon is $2a$ and O is the centre of the hexagon taken as origin. Also take OC as x - axis and OM as y-axis.
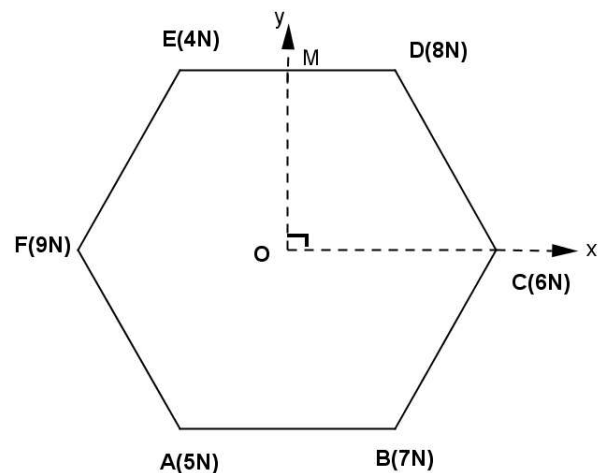
$AB = 2a \Rightarrow OC = 2a = OD$

and $OM = \sqrt{4a^2 - a^2} = \sqrt{3}a$

Let coordinates of centre of gravity be $(\bar{x}, \bar{y})$

Taking moment about OC

$$6.2a + 8.a + 7.a + 4.(-a) + 5.(-a) + 9.(-2a) = (6 + 8 + 7 + 4 + 5 + 9)\bar{x}$$
$$\bar{x} = \frac{27a - 27a}{39}$$
$$= 0$$

Taking moment about OM

$$8.a\sqrt{3} + 4.a\sqrt{3} + 6.0 + 9.0 + 5.(-a\sqrt{3}) + 7.(-a\sqrt{3}) = (6 + 8 + 7 + 4 + 5 + 9)\bar{y}$$

$$\bar{y} = \frac{12a\sqrt{3} - 12a\sqrt{3}}{39}$$

$$= 0$$

The centre of gravity coincides with the point O which is the centre of the hexagon.

Example 4

A uniform circular disc of radius $\frac{r}{2}$ is cut off from a circle of a radius r of the disc as diameter. Find the centre of gravity of the remainder.

Let AB be the diameter of the circular disc and O its centre.

Let O′ be the centre of the disc described on AO as diameter and w be the weight per unit area.

Weight of the large circular disc $= \pi r^2 w$

Weight of the small circular disc $= \pi \left(\frac{r}{2}\right)^2 w = \frac{1}{4}\pi r^2 w$

Let G be the centre of gravity of the remainder.

By symmetry centre of gravity of the remainder lies on AB.

Taking moment about AY,

$$r\left(\pi r^2 w - \frac{\pi}{4}r^2 w\right)AG = \pi r^2 w \times AO - \frac{\pi}{4}r^2 w \times AO'$$

$$= \frac{\pi r^2 w.\ r\ -\ \frac{\pi}{4}r^2 w.\frac{r}{2}}{\frac{3}{4}\pi r^2 w}$$

$$= \frac{\frac{7}{8}r}{\frac{3}{4}}$$

$$= \frac{7}{6}r$$

$$OG = \frac{7}{6}r - r = \frac{r}{6}$$
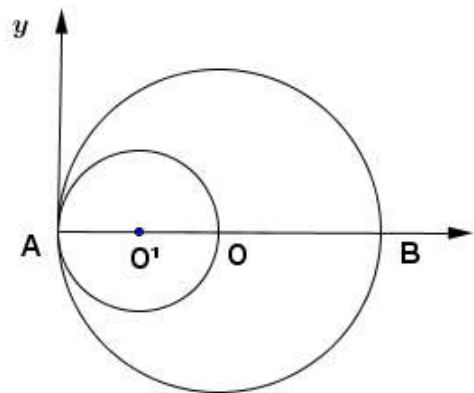
Therefore, the distance of the centre of gravity of the remainder from the centre of the original disc is $\frac{1}{6}r$ along the diameter.

Example 5

ABCD is a square lamina of side $2a$. E is the midpoint of the side BC. Find the distance of the centre of gravity of the portion AECD from A.

Let AB and AD are the x, y axes respectively and w be the weight of unit area.

Weight of lamina ABCD is $4a^2w$

Weight of portion ABE is $\dfrac{1}{2} \cdot 2a^2w = a^2w$

Let G_1, G_2 be the centre of gravity of ABCD and ABE respectively and G be the centre of gravity of the portion ABCD.

Let $G = (\overline{x}, \overline{y})$

Taking moment about AD,

$$\left(4a^2w - a^2w\right)\overline{x} = 4a^2w \times a - a^2w \times \frac{2}{3} \times 2a$$

$$3a^2w\,\overline{x} = \frac{8}{3}a^3w$$

$$\overline{x} = \frac{8}{9}a$$
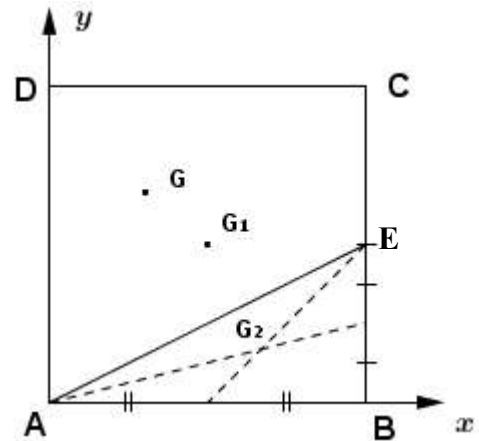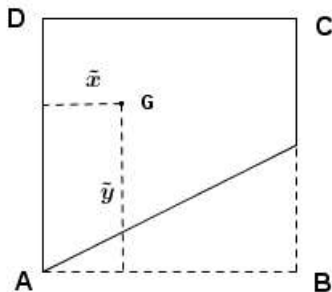
Taking moment about AB,

$$\left(4a^2w - a^2w\right)\overline{y} = 4a^2w \times a - a^2w \times \frac{1}{3} \times a$$

$$3a^2w\,\overline{y} = \frac{11}{3}a^3w$$

$$\overline{y} = \frac{11}{9}a$$

$$AG^2 = \overline{x}^2 + \overline{y}^2$$

$$= \left(\frac{8a}{9}\right)^2 + \left(\frac{11a}{9}\right)^2$$

$$= \frac{\sqrt{185}}{9}a$$

Example 6

A uniform triangular lamina ABC, obtuse angled at C stands in a vertical plane with the side AC in contact with a horizontal table. Show that the largest weight, which if suspended from vertex B will not overturn the lamina is $\dfrac{1}{3}W\left(\dfrac{a^2 + 3b^2 - c^2}{c^2 - a^2 - b^2}\right)$, where W is the weight of the triangle and a, b, c have their usual meanings.

Centre of gravity of the lamina is same as the equal weights on the vertices of the triangle.

So the weight on points A, B, C is $\dfrac{1}{3}W$

Let w be the largest weight suspended from B. At this stage reaction given by the table to the lamina acts through the point C.

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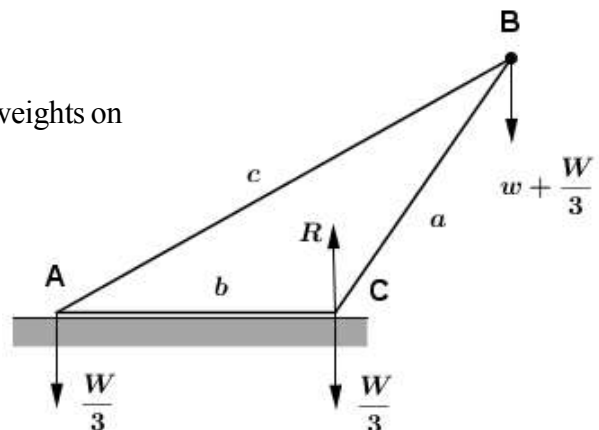
The weight W of the lamina can be considered as three particles of weight $\dfrac{W}{3}$ placed on vertices A, B and C. As the weight w increases the lamina tends to topple about point C. When w is maximum, reaction R will act at C.

For the equilibrium of the lamina taking moment about C,

$$\left(w+\dfrac{W}{3}\right)a\cos(\pi-c)-\dfrac{W}{3}.b=0$$

$$\left(\dfrac{w+\dfrac{W}{3}}{\dfrac{W}{3}}\right)=\dfrac{b}{-a\cos c}=\dfrac{b}{-a\left(\dfrac{a^2+b^2-c^2}{2ab}\right)}=\dfrac{2b^2}{c^2-a^2-b^2}$$

$$\dfrac{w}{\dfrac{W}{3}}=\dfrac{2b^2-(c^2-a^2-b^2)}{c^2-a^2-b^2}$$

$$w=\dfrac{W}{3}\left(\dfrac{3b^2+a^2-c^2}{c^2-a^2-b^2}\right)$$

Centre of gravity of a circular arc

Let AB be a circular arc and O be the centre of the circle whose radius is a. AB subtends angle 2α at the centre O.

P, Q are the two close points on the arc such that $P\hat{O}Q=\delta\theta$ and $M\hat{O}P=\theta$

M be the midpoint of the arc, $M\hat{O}P=\theta$, w weight per unit length

Weight of elemet PQ $= a\ \delta\theta\ w$

Weight of the arc AB $=\displaystyle\int_{-\infty}^{+\infty}a\ d\theta\ w$

Centre of gravity of element PQ from O is $a\cos\theta$.

By symmetry centre of the arc AB lies on OM.

Let G be the centre of gravity of the arc AB.

Taking moments about O

$$\left[\int_{-\alpha}^{+\alpha}a\ d\theta\ w\right]OG=\int_{-\alpha}^{+\alpha}a\ d\theta\ w\,.\,a\cos\theta$$

$$aw\int_{-\alpha}^{+\alpha}d\theta\,.OG=a^2w\int\cos\theta\ d\theta$$

$$aw\left[\theta\right]_{-\alpha}^{+\alpha}.OG=a^2w\left[\sin\theta\right]_{-\alpha}^{+\alpha}$$

$$aw\,.\,2\alpha\,.\,OG=a^2w\,.\,2\sin\alpha$$

$$OG=\dfrac{a\sin\alpha}{\alpha}$$

Deduction :

Centre of gravity of semicircular arc, when $\alpha=\dfrac{\pi}{2}$, $OG=\dfrac{a\sin\dfrac{\pi}{2}}{\dfrac{\pi}{2}}=\dfrac{2a}{\pi}$
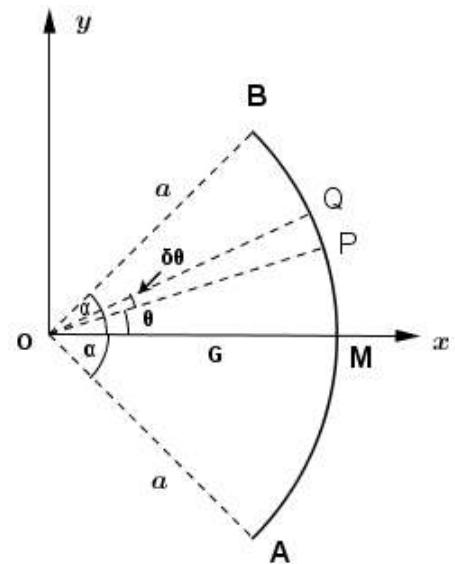
Center of gravity of a sector of a circle

Let AOB be a sector of a circle of radius a and centre O.

Arc AB subtends 2α at O. M be the midpoint of AB.

P, Q are two close point on the arc AB such that $\hat{MOP} = \theta$ and

$\hat{POQ} = \delta\theta$. Let m be the weight per unit area

weight of $\triangle$ POQ $= \dfrac{1}{2}a^2\delta\theta\ m$

weight of sector AOB $= \displaystyle\int_{-\alpha}^{+\alpha} \dfrac{1}{2}a^2 d\theta\ m$

Centre of distance of AOB from O is $\dfrac{2}{3}a\cos\theta$

By symmetry centre of gravity of the sector G lies on OM

Taking moments about O,

$$\left[\int_{-\alpha}^{+\alpha}\dfrac{1}{2}a^2 d\theta.m\right]OG = \int_{-\alpha}^{+\alpha}\dfrac{1}{2}a^2 d\theta.m.\dfrac{2}{3}a\cos\theta$$

$$\dfrac{ma^2}{2}[\theta]_{-\alpha}^{+\alpha}.OG = \dfrac{ma^3}{3}[\sin\theta]_{-\alpha}^{+\alpha}$$

$$\dfrac{ma^2}{2}[2\alpha].OG = \dfrac{ma^3}{3}.2\sin\alpha$$

$$OG = \dfrac{2}{3}.a\dfrac{\sin\alpha}{\alpha}$$
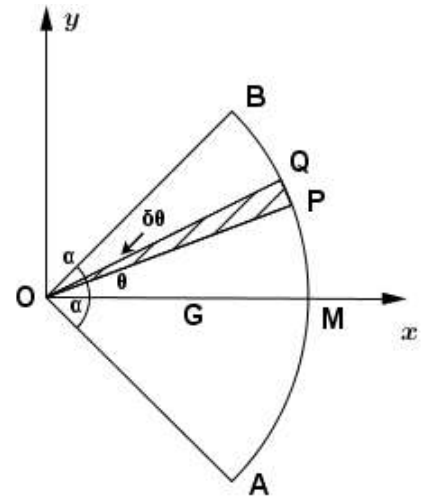
Deduction :
Centre of gravity of a semicircular disc.

when $\alpha = \dfrac{\pi}{2}$, $OG = \dfrac{2}{3}\dfrac{a\sin\dfrac{\pi}{2}}{\dfrac{\pi}{2}} = \dfrac{4a}{3\pi}$

Centre of gravity of a segment of a circle

Let AMB is a segment of a circle with centre O and radius a.

By symmetry centre of gravity of the segment G lies on OM.

w - weight of a unit area

Figure	Weight	Centre of gravity from O
Sector OAMB	$\dfrac{1}{2}a^2.2\alpha.w$	$\dfrac{2}{3}a\dfrac{\sin\alpha}{\alpha}$
Triangle OAB	$\dfrac{1}{2}.2a\sin\alpha\ .\ a\cos\alpha.\ w$	$\dfrac{2}{3}a\cos\alpha$
Segment AMB	$a^2(\alpha - \sin\alpha\cos\alpha)w$	OG
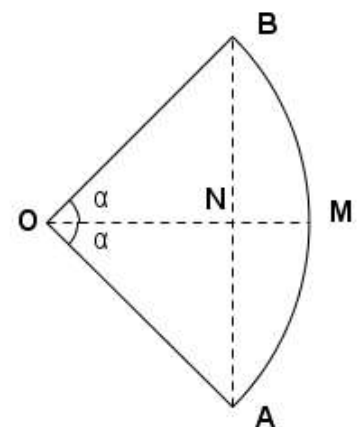
Taking moment about O,

$$a^2(\alpha - \sin\alpha\cos\alpha)w.\text{OG} = \frac{1}{2}a^2.2\alpha.w.\frac{2}{3}a\frac{\sin\alpha}{\alpha} - \frac{1}{2}.2a\sin\alpha.a\cos\alpha.w.\frac{2}{3}a\cos\alpha$$

$$(\alpha - \sin\alpha\cos\alpha)w.\text{OG} = \frac{2}{3}a\sin\alpha - \frac{2}{3}a\sin\alpha\cos^2\alpha$$

$$= \frac{2}{3}a\sin\alpha(1-\cos^2\alpha)$$

$$= \frac{2}{3}a\sin^2\alpha$$

$$\text{OG} = \frac{2a\sin^3\alpha}{3(\alpha - \sin\alpha\cos\alpha)}$$

Deduction :

When $\alpha = \dfrac{\pi}{2}$, segment becomes a semicircular lamina, $\text{OG} = \dfrac{4a}{3\pi}$

Centre of gravity of a solid hemisphere

Let OM be the axis of symmetry and O is the centre and a the radius of the sphere.

Let PQ be circular disc with thickness δx and in a distance x from O

Let w be the density of the sphere.

Mass of $\text{PQ} = \pi r^2 \delta x.w$

Centre of gravity of PQ, $\pi(a^2 - x^2)\delta x$ w from O is x.

$$\therefore \text{ Mass of the hemisphere} = \int_0^a \pi\left(a^2 - x^2\right)dx \text{ w}$$

By symmetry, centre of gravity of the hemisphere G lies on OM.

Taking moments about O,

$$\left[\int_0^a \pi\left(a^2 - x^2\right)dx.w\right]\text{OG} = \int \pi\left(a^2 - x^2\right)dx.w.x$$

$$\pi w\left[a^2 x - \frac{x^3}{3}\right]_0^a \text{OG} = \pi w\left[\frac{a^2 x^2}{2} - \frac{x^4}{4}\right]_0^a$$

$$\pi w \times \frac{2}{3}a^3.\text{OG} = \pi w.\frac{a^4}{4}$$

$$\text{OG} = \frac{3}{8}a$$
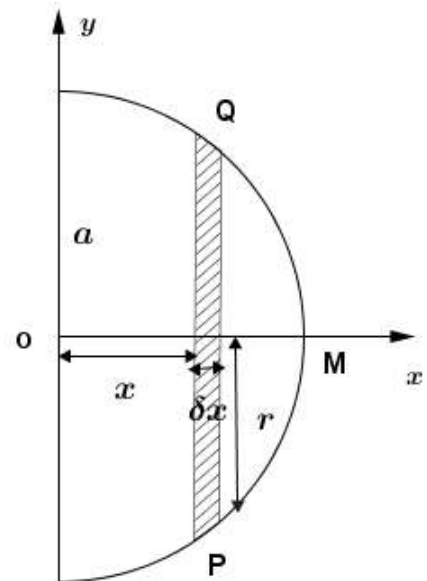
Centre of gravity of a hollow hemisphere

Let OM be the axis of symmetry, O the centre and a the radius of the sphere.

Let PQ be circular ring with the height of a $\delta\theta$ and in a distance $a\cos\theta$ from O.

Also let w be the weight per unit area.

weight of PQ $= (2\pi a\delta\theta)(a\delta\theta).w$

Centre of gravity of PQ from O is $a\cos\theta$.

$\therefore$ Mass of the hollow hemisphere $= \int_0^{\frac{\pi}{2}} 2\pi a \sin\theta \, a \, d\theta \, w$

By symmetry centre of gravity of the hemi sphere G lies on OM

Taking moments about O,

$$\left[\int_0^{\frac{\pi}{2}} 2\pi a \sin\theta \, a \, d\theta \, w\right] OG = \int_0^{\frac{\pi}{2}} 2\pi a \sin\theta \, a \, d\theta \, wa\cos\theta$$

$$2\pi a^2 w\int_0^{\frac{\pi}{2}} \sin\theta \, d\theta . \, OG = \pi a^3 w\int_0^{\frac{\pi}{2}} \sin 2\theta \, d\theta$$

$$2\pi a^2 w\left[-\cos\theta\right]_0^{\frac{\pi}{2}} .OG = \pi a^3 w\left[\frac{-\cos 2\theta}{2}\right]_0^{\frac{\pi}{2}}$$

$$2\pi a^2 w\left[0-(-1)\right] .OG = \pi a^3 w$$

$$OG = \frac{a}{2}$$
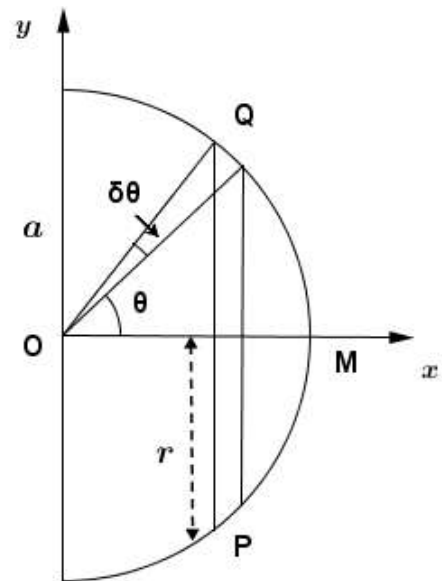
Centre of gravity of a solid cone

Let h be the height and α be semi vertical angle of the cone.

Consider a circular disc PQ with thickness δx and a distance x from vertex O.

Let w be density of the cone

Weight of PQ $= \pi r^2 \delta x$ w g

$= \pi(x\tan\alpha)^2 \, \text{f} x.$w g

Weight of the cone $= \int_0^{h} \pi x^2 \tan^2\alpha.dx$ w.g

Centre of gravity of PQ from O is x

By symmetry centre of gravity of the cone G lies on OM.

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Taking moment about O

$$OG \cdot \left[\int_0^h \pi x^2 \tan^2 \alpha \; dx \; w\right] = \int_0^h \pi x^2 \tan^2 \alpha \; dx \; w.x$$

$$OG \cdot \left[\pi \tan^2 \alpha \; w \int_0^h x^2 dx\right] = \pi \tan^2 \alpha \; w \int_0^h x^3 dx$$

$$OG \cdot \pi \tan^2 \alpha \; w \left.\frac{x^3}{3}\right|_0^h = \pi \tan^2 \alpha \; w \left.\frac{x^4}{4}\right|_0^h$$

$$OG \cdot \frac{\pi}{3} h^3 \tan^2 \alpha \; w = \frac{\pi}{4} h^4 \tan^2 \alpha \; w$$

$$\therefore \; OG = \frac{3}{4} h$$

Centre of gravity of a hollow cone

Let h be the height and α be semi-vertical angle of the cone.

Consider a circular ring PQ with a height δx at a distance $x \cos \alpha$

Let w be the weight per unit area.

Weight of PQ $= 2\pi (x \sin \alpha) \delta x.w$

Weight of the Cone $= \int_0^\ell 2\pi (x \sin \alpha) dx.w$

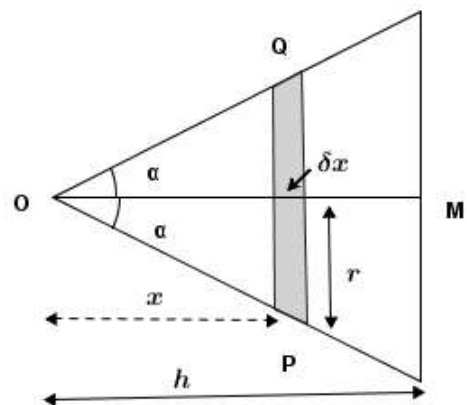By symmetry centre of gravity of the cone G lies on OM

Taking moments about O,

$$OG \cdot \left[\int_0^\ell 2\pi(x \sin \alpha) \; dx.w\right] = \int_0^\ell 2\pi x \sin \alpha \; dx \cdot x \cos \alpha.w$$

$$OG.2\pi \sin\alpha.w \int_0^\ell x \; dx = 2\pi \sin \alpha \cos \alpha \int x^2 dx.w$$

$$OG.2\pi\sin\alpha.w \left.\frac{x^2}{2}\right|_0^\ell = 2\pi\sin\alpha \cos \alpha.w \left.\frac{x^3}{3}\right|_0^\ell$$

$$OG.\left[2\pi\sin\alpha \; w \; \frac{\ell^2}{2}\right] = 2\pi \sin \alpha \cos\alpha.w \frac{\ell^3}{3}$$

$$OG = \frac{2}{3} \ell \cos\alpha$$
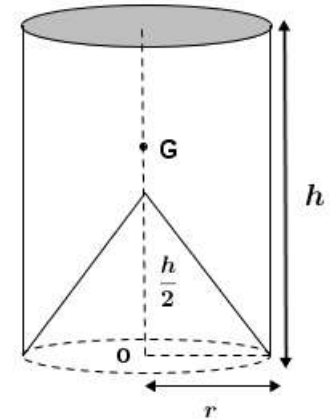
$$OG = \frac{2}{3} h$$
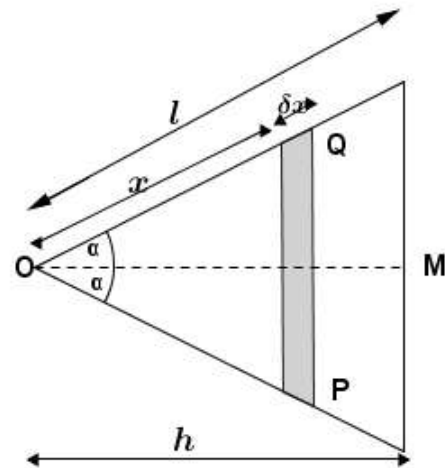
Example 7

From a uniform solid right circular cylinder of radius r and height h, a solid right circular cone of radius r and height $\dfrac{h}{2}$ is bored out so that the base of the cone coincides with one end of the cylinder. Show that the centre of gravity of the remainder is on the axis at a distance $\dfrac{23h}{40}$ from the base of the cone.

By symmetry, centre of gravity of the remainder lies on the axis through O.

Figure	Weight	Centre of Gravity from O
Cylinder	$\pi r^2 h \rho g$	$\dfrac{h}{2}$
Cone	$\dfrac{1}{3}\pi r^2 \dfrac{h}{2}\rho g$	$\dfrac{1}{4}\left(\dfrac{h}{2}\right)=\dfrac{h}{8}$
Remainder	$\dfrac{5}{6}\pi r^2 h\rho g$	OG

Taking moment about O,

$$\frac{5}{6}\pi r^2 h\rho g.\text{OG} = \pi r^2 h\rho g\left(\frac{h}{2}\right) - \frac{1}{3}\pi r^2\left(\frac{h}{2}\right)\rho g.\left(\frac{h}{8}\right)$$

$$\frac{5}{6}.\text{OG} = \frac{h}{2} - \frac{h}{48} = \frac{23h}{48}$$

$$\text{OG} = \frac{23h}{40}$$

Example 8

A uniform solid body formed by welding together at coincidental bases of radii a, a hemisphere and a right circular cone of semi-vertical angle α. If the body can rest in equlibrium with any point of the curved surface of the hemisphere in contact with a horizontal table, find the value of α.

The body rest in equlibrium with any point of the curved surface of the hemisphere in contact.

Then the reaction through the point of contact and the weight of the whole body $(w_1 + w_2)$ should act through the point of contact and the centre O.

Therefore, taking moment about O
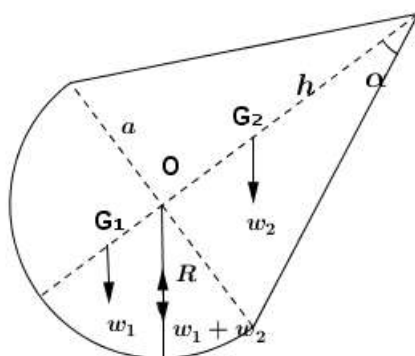
$$w_1.\,\text{OG}_1 - w_2.\text{OG}_2 = 0$$

$$\frac{2}{3}\pi a^3\rho.\frac{3}{8}a - \frac{1}{3}\pi a^2 h\rho.\frac{1}{4}h = 0$$

$$3a^2 = h^2$$

$$\frac{a}{h} = \frac{1}{\sqrt{3}}$$

$$\tan\alpha = \frac{1}{\sqrt{3}}$$

$$\alpha = \frac{\pi}{6}$$

Example 9

A toy is in the form of a composite body formed by joining together a uniform solid right circular cone of density ρ, base radius a and height $4a$ and a uniform solid hemisphere of density $\lambda\rho$ and base radius a so that their bases coincide. Find the distance of the centre of gravity of the toy from the centre of the common base. If the toy cannot be in stable equilibrium with the curved surface of the cone in contact with a smooth horizontal plane, show that $\lambda > 20$.
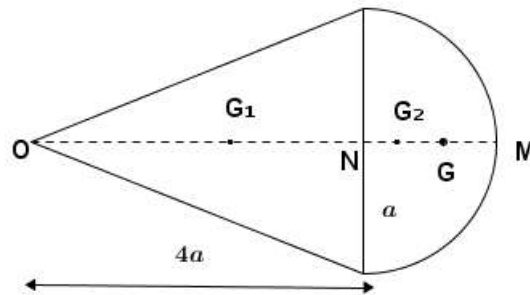
By symmetry centre of gravity of the toy G lies on OM.

Figure	Weight	Centre of gravity from N
Cone	$\dfrac{1}{3}\pi a^2 . 4a . \rho g$	$NG_1 = -\dfrac{1}{4}.4a = -a$
Hemisphere	$\dfrac{2}{3}\pi a^3 . \lambda\rho g$	$NG_2 = \dfrac{3a}{8}$
Toy	$\dfrac{2}{3}\pi a^3 \rho(2+\lambda) g$	NG

Taking moment about O

$$\frac{2}{3}\pi a^3 \rho(2+\lambda)g \, . \, NG = \frac{4}{3}\pi a^3 \rho g(-a) + \frac{2}{3}\pi a^3 \lambda\rho \; g . \frac{3a}{8}$$

$$(2+\lambda).NG = -2a + \frac{3a}{8}\lambda$$

$$NG = \frac{(3\lambda - 16)}{8(2+\lambda)}a$$

For non stability $NC < NG$

$$a\tan\alpha < \frac{(3\lambda-16)}{8(2+\lambda)}a$$

$$\frac{1}{4} < \frac{(3\lambda-16)}{8(2+\lambda)}$$

$$2(2+\lambda) < 3\lambda - 16$$

$$20 < \lambda$$

Example 10

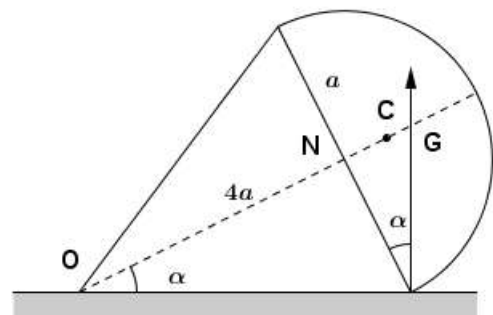A uniform solid cone of semi - vertical angle 15° rests with its base on a rough horizontal floor. It is tilted to one side by a light inextensible string attached to its vertex. The string pulls the cone downwards making an angle 45° with the horizontal, in a vertical plane through the axis of the cone. The edge of the cone is about to slip on the floor, when the vertex is vertically above the point of contact of the edge and the floor. Write down the sufficient equations to determine the tension T in the string, the normal reaction and the frictional force. Hence show that

i. $T = \dfrac{3\sqrt{2}}{16} W$

ii. The value of the coefficient of friction is $\dfrac{3}{19}$

For the equilibrium of the cone

Taking moment about A

$$Am \qquad T.\ell \sin 45° - W.\frac{3}{4} h \sin 15° = 0$$

$$T.h \sec 15°. \sin 45° - W.\frac{3}{4} h \sin 15° = 0$$

$$\frac{T}{\sqrt{2}\cos 15°} = \frac{3}{4} W \sin 15°$$

$$T = \frac{3\sqrt{2}}{8} W \sin 30°$$

$$= \frac{3\sqrt{2}}{16} W$$
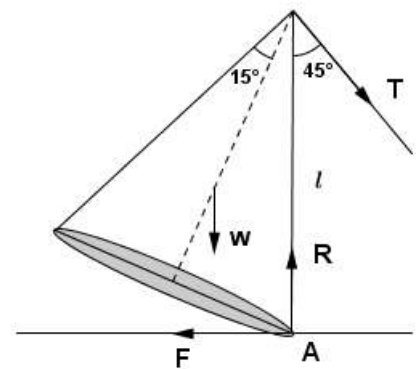
Resolving vertically,

$\uparrow$ R. T$\cos$ 45° - W = 0

$$R = \frac{19}{16} W$$

Resolving horizontally,

$\leftarrow$ F - T$\sin$ 45° = 0

$$F = \frac{3}{16} W$$

For limiting equilibrium,

$$\frac{F}{R} = \mu$$

$$\mu = \frac{\dfrac{3}{16} W}{\dfrac{19}{16} W}$$

$$= \frac{3}{19}$$

Example 11

A solid is formed by removing a right circular cone of base radius a and height a from a uniform solid hemisphere of radius a. The plane base of the hemisphere and the cone are coincidental with O as the common centre of both. Find the distance from O of the centre of the mass G of the solid.

The figure show the cross section of the above solid resting in equilibrium with a point on the curved surface in contact with a rough plane inclined at angle θ to the horizontal. O and G are in same vertical plane through a line of greatest slope of the plane. Given that OG is horizontal. Show that $\theta = 30^0$. Given the weight of the hemisphere is W.

Obtain in terms of W the values of the frictional force and the normal reaction at the point of contact.

Find also the smallest value of the coefficient of friction between the plane and the solid.
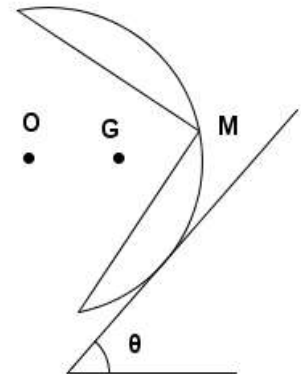

By symmetry centre of gravity of the remainder a lies on the axis of the cone.

Figure	Weight	Centre of Gravity from O
Hemisphere	$\frac{2}{3}\pi a^3 W$	$\frac{3}{8}a$
Cone	$\frac{1}{3}\pi a^2.aW$	$\frac{1}{4}a$
Remainder	$\frac{1}{3}\pi a^3 W$	OG

Taking moment about O,

$$\frac{1}{3}\pi a^3 W . OG = \frac{2}{3}\pi a^3 W . \frac{3}{8}a - \frac{1}{3}\pi a^3 W . \frac{1}{4}a$$

$$OG = \frac{6}{8}a - \frac{1}{4}a$$

$$= \frac{a}{2}$$

For the equilibrium of the solid, weight of the solid $\frac{W}{2}$ should pass
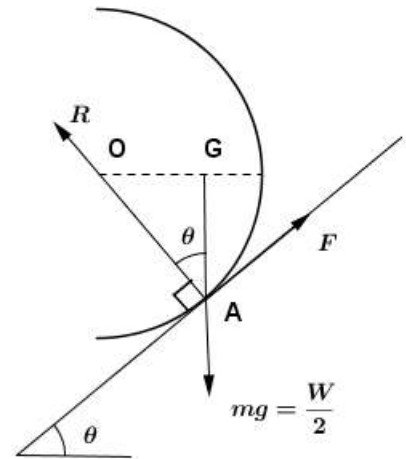
lines of the action of three forces F, R, $\frac{W}{2}$ through A

$$\therefore \sin\theta = \frac{OG}{OA}$$

$$= \frac{\frac{a}{2}}{a}$$

$$= \frac{1}{2}$$

$$\theta = 30°$$

Resolving parallel to the plane,

$$\nearrow \quad F - \frac{W}{2}\sin\theta = 0$$

$$F = \frac{W}{2}\sin\theta$$

$$= \frac{W}{2}\sin 30°$$

$$= \frac{W}{4}$$

Resolving perpendicular to the plane,

$$\nwarrow \quad R - \frac{W}{2}\cos\theta = 0$$

$$R = \frac{W}{2}\cos\theta$$

$$= \frac{W}{2}\cos 30°$$

$$= \frac{W\sqrt{3}}{4}$$

For equilibrium,

$$\frac{F}{R} \le \mu$$

$$\frac{\frac{W}{4}}{\frac{W\sqrt{3}}{4}} \le \mu$$

$$\frac{1}{\sqrt{3}} \le \mu$$

$$\mu_{min} = \frac{1}{\sqrt{3}}$$

Example 12

The figure shows the remains of a uniform solid right circular cylinder ABCD of height H and base radius R, after solid right circular cone EAB of height h and base radius R is scooped out. Find the distance of gravity of the resulting body S from AB. Hence show that if the centre of gravity of S is at E then

$$h = \left(2 - \sqrt{2}\right)H.$$

The body S is placed on a rough plane making an angle $\alpha\left(<\dfrac{\pi}{2}\right)$ with the horizontal, the base DC being on the plane. The plane is rough enough to prevent S from skipping. Assuming that centre of gravity of S is at E show that S will not topple if $R\cot\alpha > \left(\sqrt{2}-1\right)H$.

By symmetry centre of gravity of S lies on the axis of the cylinder.

Figure	Weight	Centre of gravity from A	
Cylinder	$\pi R^2 HW$	$\dfrac{H}{2}$	
Cone	$\dfrac{1}{3}\pi R^2 hW$	$\dfrac{h}{4}$	
Body S	$\pi R^2\left(H - \dfrac{h}{3}\right)W$	$\bar{y}$	
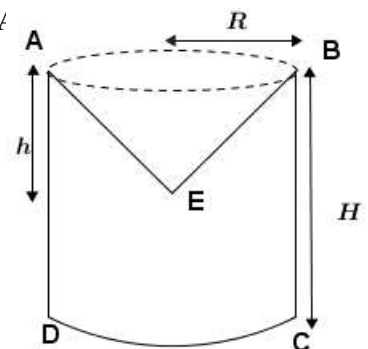
Taking moment about AB

$$\pi R^2 \left(H - \frac{h}{3} \right) W \; \overline{y} = \pi R^2 H \; W \cdot \frac{H}{2} - \frac{1}{3} \pi R^2 h W \cdot \frac{1}{4} h$$

$$\left(H - \frac{h}{3} \right) \overline{y} = \frac{H^2}{2} - \frac{h^2}{12}$$

$$\overline{y} = \frac{6H^2 - h^2}{4(3H - h)}$$

If centre of gravity is on E, then $\overline{y} = h$

$$h = \frac{6H^2 - h^2}{4(3H\text{-}h)}$$

$$\Rightarrow 3h^2 - 12Hh + 6H^2 = 0$$

$$h^2 - 4Hh + 2H^2 = 0$$

$$(h - 2H)^2 - 2H^2 = 0$$

$$\left(h - 2H + \sqrt{2}H \right)\left(h - 2H - \sqrt{2}H \right) = 0$$

$$h = 2H - \sqrt{2}H, \quad 2H + \sqrt{2}H$$

$$\Rightarrow h < H \Rightarrow h = 2H - \sqrt{2}H$$

$$= \left(2 - \sqrt{2} \right) H$$

If KM < DM, the body will not topple.

$$(H - h)\tan \alpha < R$$

$$(H - h) < R \cot \alpha$$

$$\left(\sqrt{2} - 1 \right) H < R \cot \alpha$$
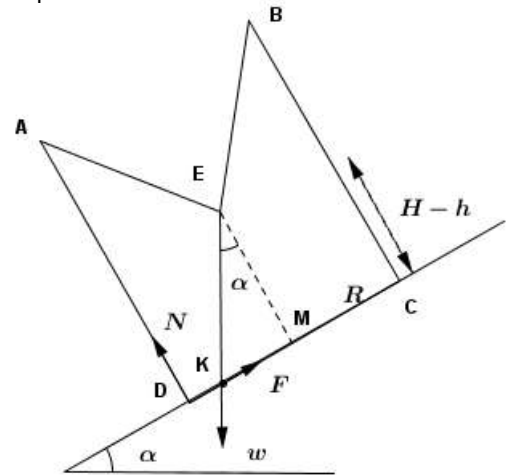
Example 13

In the figure below, ABCD represents a uniform solid body of density ρ in the form of a frustum of height h of a right circular cone. The diameters of its circular plane faces are $AB = 2a\lambda$, and $CD = 2a$ where λ is a parameter and $0 < \lambda < 1$.

Show by intergration that its mass is $\frac{1}{3}\pi a^2 h \rho (1 + \lambda + \lambda^2)$ and that its centre of mass G is at a distance

$\frac{h}{4}\left(\frac{3 + 2\lambda + \lambda^2}{1 + \lambda + \lambda^2} \right)$ from the centre of the smaller face.

Deduce the mass and the position of the centre of mass of uniform right circular solid cone of base radius a and height h.

A solid body J is obtained frustum ABCD by scooping out a right

circular solid cone VAB of base radius λa and hight $\frac{h}{2}$.

Find the position of the centre of mass G_1 of J and verify that G_1 cannot coincide with V.
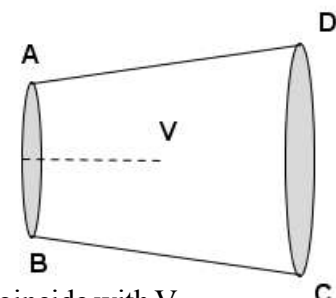
The body J is suspended freely from a point on the circumference of the larger face. Show that in the position of equilibrium the axis of symmetry of J makes and acute angle β with the vertical given by

$$\tan \beta = \frac{8a}{h}\left(\frac{2+2\lambda+\lambda^2}{4+8\lambda+5\lambda^2}\right)$$

$$\frac{x}{r-\lambda a} = \frac{h}{a(1-\lambda)}$$

$$r-\lambda a = \frac{a(1-\lambda)}{h}x$$

$$r = \frac{a(1-\lambda)}{h}x + \lambda a$$

Consider circular disc PQ with height of δx at a distance x from AB.

Volume of PQ $= \pi r^2 \delta x$

Mass of PQ $= \pi r^2 \delta x \rho$

Mass of the frustum $= \int_0^h \pi r^2 dx \rho$

$$\therefore \qquad = \int_0^h \pi.\left[\frac{a(1-\lambda)x}{h}+\lambda a\right]^2 dx\rho = \pi\rho\left|\frac{\left[\frac{a(1-\lambda)x}{h}+\lambda a\right]^3}{3a\frac{(1-\lambda)}{h}}\right|_0^h$$
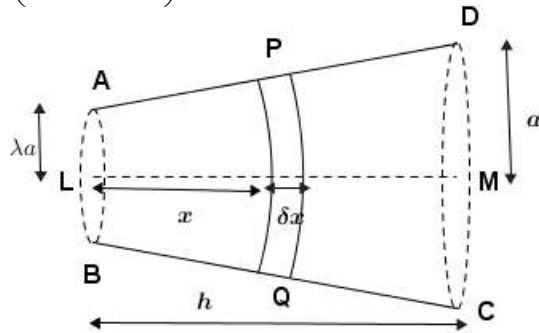
$$= \frac{\pi\rho}{3}\frac{h}{a(1-\lambda)}\left\{\left[a(1-\lambda)+\lambda a\right]^3 - (\lambda a)^3\right\}$$

$$= \frac{\pi\rho}{3}\frac{ha^3(1-\lambda^3)}{a(1-\lambda)} = \frac{\pi}{3}a^2 h\rho\frac{(1-\lambda^3)}{(1-\lambda)}$$

$$M = \frac{\pi}{3}a^2 h(1+\lambda+\lambda^2)\rho \dots\dots\dots\dots\dots(1)$$

By symmetry centre of mass G lies on the line connecting the centre of bases.

$$M.\ LG = \int \pi r^2 \delta x \rho.x$$

$$LG = \frac{\int_0^h \pi\rho\left[\frac{a(1-\lambda)}{h}x+\lambda a\right]^2 xdx}{M}$$

$$= \frac{\pi\rho}{M}\int_0^h\left[\frac{a^2(1-\lambda)^2}{h^2}x^3 + \frac{2\lambda a^2}{h}(1-\lambda)x^2 + \lambda^2 a^2 x\right]dx$$

$$= \frac{\pi\rho}{M}\left[\frac{a^2(1-\lambda)^2}{h^2}\frac{x^4}{4} + 2\lambda\frac{a^2}{h}(1-\lambda)\left(\frac{x^3}{3}\right) + \lambda^2 a^2\frac{x^2}{2}\right]_0^h$$

$$= \frac{\pi\rho}{M}\left[\frac{a^2(1-\lambda)^2}{h^2}\frac{x^4}{4} + 2\lambda\frac{a^2}{h}(1-\lambda)\left(\frac{x^3}{3}\right) + \lambda^2 a^2\frac{x^2}{2}\right]$$

$$= \frac{\pi\rho}{M}h^2 a^2\left[\frac{(1-\lambda)^2}{4} + \frac{2}{3}\lambda(1-\lambda) + \frac{\lambda^2}{2}\right]$$

$$= \frac{\pi a^2 h^2 \rho}{M}\left[\frac{3(1-2\lambda+\lambda^2)+8\lambda+8\lambda^2+6\lambda^2}{4\times3}\right]$$

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$$= \frac{\pi a^2 h^2 \rho}{12M}\left(\lambda^2 + 2\lambda + 3\right)$$

$$= \frac{\pi a^2 h^2 \rho}{12} \cdot \frac{\left(\lambda^2 + 2\lambda + 3\right)}{\frac{\pi}{3} a^2 h\left(1 + \lambda + \lambda^2\right)\rho}$$

$$= \frac{h}{4}\left(\frac{\lambda^2 + 2\lambda + 3}{\lambda^2 + \lambda + 1}\right) \qquad \text{...........................(2)}$$

When $\lambda = 0$ the frustum becomes a cone with height of h and base radius a.

$\therefore$ Mass of the cone $= \frac{1}{3}\pi a^2 h\rho$ from (1) with $\lambda = 0$

Centre of the mass of the cone from the vertex is $\frac{h}{4}\cdot\frac{3}{1} = \frac{3h}{4}$ [from (2) of $\lambda = 0$]

Finding the centre of gravity of J

Figure	Weight	Centre of gravity from AB
Frustum	$\frac{1}{3}\pi a^2 \rho g(1+\lambda+\lambda^2)h$	$\frac{h}{4}\left(\frac{\lambda^2 + 2\lambda + 3}{\lambda^2 + \lambda + 1}\right)$
Cone VAB	$\frac{1}{3}\pi\left(\lambda a\right)^2 \rho g \frac{h}{2}$	$\frac{1}{4}\cdot\frac{h}{2} = \frac{h}{8}$
Remainder	$\frac{1}{3}\pi a^2 h\rho g\left(1 + \lambda + \frac{\lambda^2}{2}\right)$	$\bar{y}$

Taking moment about L

$$\frac{1}{3}\pi a^2 h\rho g\left(1 + \lambda + \frac{\lambda^2}{2}\right)\bar{y} = \frac{1}{3}\pi a^2 \rho g\left(1 + \lambda + \lambda^2\right)\cdot\frac{h}{4}\left(\frac{\left(\lambda^2 + 2\lambda + 3\right)}{\lambda^2 + \lambda + 1}\right) - \frac{1}{3}\pi a^2 \lambda^2 \rho \frac{h}{2} g\cdot\frac{h}{8}$$

$$\left(1 + \lambda + \frac{\lambda^2}{2}\right)\bar{y} = \frac{h}{4}\left(\lambda^2 + 2\lambda + 3\right) - \frac{h\lambda^2}{16}$$

$$\bar{y} = \frac{h}{8}\left(\frac{3\lambda^2 + 8\lambda + 12}{\lambda^2 + 2\lambda + 2}\right)$$

$$\bar{y} - \frac{h}{2} = \frac{h}{8}\left(\frac{3\lambda^2 + 8\lambda + 12}{\lambda^2 + 2\lambda + 2}\right) - \frac{h}{2}$$

$$= \frac{h}{8}\left[\frac{3\lambda^2 + 8\lambda + 12 - 4\left(\lambda^2 + 2\lambda + 2\right)}{\lambda^2 + 2\lambda + 2}\right]$$

$$= \frac{h}{8}\left(\frac{4 - \lambda^2}{\lambda^2 + 2\lambda + 2}\right) > 0 \qquad (\because 0 < \lambda < 1)$$
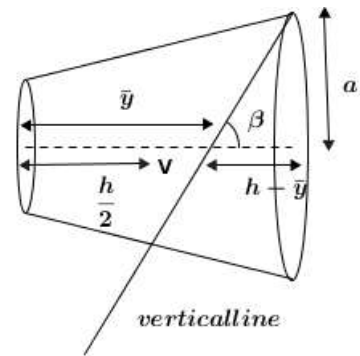
$\therefore$ Point V cannot coincide with G_1

$$\tan\beta = \frac{a}{h - \bar{y}}$$

$$h - \bar{y} = h - \frac{h}{8}\left(\frac{3\lambda^2 + 8\lambda + 12}{2 + 2\lambda + \lambda^2}\right)$$

$$= \frac{h}{8}\left(\frac{5\lambda^2 + 8\lambda + 4}{2 + 2\lambda + \lambda^2}\right)$$

$$\therefore \tan\beta = \frac{8a}{h}\left(\frac{2 + 2\lambda + \lambda^2}{4 + 8\lambda + 5\lambda^2}\right)$$

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8.4 Exercises

1. From a uniform triangle ABC, a portion ADE is removed where DE is parallel to BC and the area of the triangle ADE equals to half of ABC. Find the centre of gravity of the remainder from BC.

2. From a triangle ABC, a portion ADE, where DE is parallel to BC, is removed. If a and b be the distances of A from BC and DE respectively, show that the distance of the centre of gravity of the remainder from BC is $\dfrac{a^2 + ab - 2b^2}{3(a+b)}$.

3. Three rods of length a, b, c are joined at their ends so as to form a triangle. Find the centre of gravity of the triangle.

4. From a uniform triangular board ABC a portion consisting of the area of the inscribed circular is removed. Show that the distance of the centre of gravity of the remainder from BC is $\dfrac{S}{3as}\left[\dfrac{2s^3 - 3\pi a S}{s^2 - \pi S}\right]$ where S is the area, s the semi-perimeter of the board and BC $= a$.

5. ACB is a uniform semicircular lamina with diameter AOB, and OC is the radius perpendicular to AB. A square portion OPQR is cut off from the lamina, P being on OB and length of OP is $\dfrac{1}{2}a$. Find the distance from OA and OC of the centre of gravity of the remaining portion. Hence show that if the remaining portion is suspended from A and hangs in equilibrium, the tangent of the angle made by AB with the vertical is just less than $\dfrac{1}{2}$.

6. ABCDEF is a sheet of thin cardboard in the form of a regular hexagon. Prove that if the triangle ABC is cut off and superposed on the triangle DEF, the centre of gravity of the whole is moved by a distance $\dfrac{2a}{9}$, where a is the side of the hexagon.

7. Prove that the centre of gravity of a uniform semicircular lamina of radius a is at a distance $\dfrac{4a}{3\pi}$ from its centre.
   AOB is the base of a uniform semicircular lamina of radius $2a$, O being its centre. A semicircular lamina of radius a and base AO is cut away and the remainder is suspended freely from A. Find the inclination of AOB to the vertical in the equilibrium position.

8. A solid cylinder and a solid right circular cone have their bases joined together, the bases being of the same size. Find the ratio of the height of the cone to the height of the cylinder so that the centre of gravity of the compound solid may be at the centre of the common base.

9. A solid in the form of a right circular cone has its base scooped out, so that the hollow formed is a right circular cone on the same base. How much must be scooped out so that the centre of gravity of the remainder may coincide with the vertex of the hollow cone.

10. From a uniform solid right circular cone of vertical angle 60° is cut out the greatest possible sphere. Show that centre of gravity of the remainder divides the axis in the ratio 11 : 49.

11. A solid right circular cone of height h is cut off at a height $\frac{1}{2}h$ by a plane perpendicular to the axis. Find the centre of the gravity of the portion between this section and the base of the cone.

12. A hollow vessel made of uniform material of negligible thickness is in the form of a right circular cone of surface density ρ mounted on a hemisphere of surface density σ whose radius is equal to that of the circular rim of the cone. If the vessel can just rest with a generator of the cone in contact with a smooth horizontal plane, prove that semi-vertical angle a of the cone is given by the equation $\rho\left(\cot^2 \alpha + 3\right) = 3\sigma\left(\cos\alpha - 2\sin\alpha\right)$.

13. A hollow baseless cone of vertex O, semi-vertical angle α and height h is made of a uniform thin metal sheet of mass σ per unit area. Show that its mass is $\pi\sigma h^2 \sec\alpha \tan\alpha$ and find the position of its centre of mass.

    A uniform circular disc of centre B and radius $h\tan\alpha$ made of the same metal sheet is now fixed as the base of the above cone. Show that the distance of the centre of mass of the composite body from O

    is $\dfrac{h\left(\dfrac{2}{3}\sec\alpha + \tan\alpha\right)}{\sec\alpha + \tan\alpha}$.

    The composite body is suspended from a point A on the rim of the base. If AO and AB make equal angles with the downward vertical show that $\sin\alpha = \dfrac{1}{3}$.

14. A crescent shaped uniform lamina is bounded by a semicircle with centre O and radius a and a circular arc subtending an angle $\dfrac{2\pi}{3}$ at its centre C as shown in the figure. Show that the centre of mass of this lamina is at a distance ka from C, where $k = \dfrac{3\sqrt{3}\pi}{\pi + 6\sqrt{3}}$
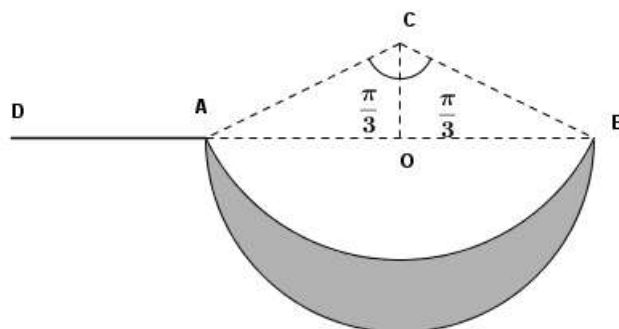
Let M be the mass of the lamina. The end A of a thin uniform straight rod AD of length $2a$ and mass m is rigidly fixed to the crescent at A along the extended line BA, forming a sickle as show in the figure. The sickle is then placed on a horizontal floor with the pan of lamina vertical and the semicircle and the free end D of the rod touching the floor. If it stays in equilibrium in this position show that $M\left(\sqrt{3}k - 1\right) < 4\sqrt{6}\, m$.
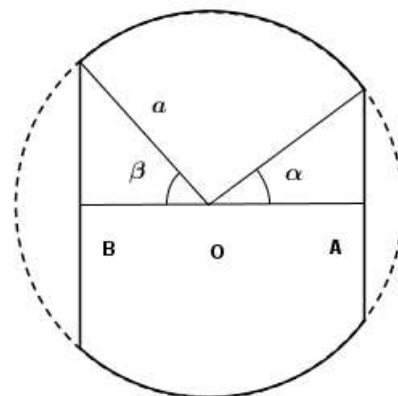
15. Out of a uniform spherical shell of radius a, centre O, and surface density σ, a cone is cut off by two parallel planes at distances $a\cos\alpha, a\cos\beta$ from O (on either side of O) where $0 < \alpha < \beta < \dfrac{\pi}{2}$ as shown in the figure.

    Show by intergration, that

    (i)   The mass of the is cone is $2\pi a^2 \sigma (\cos\alpha + \cos\beta)$

    (ii) The centre of mass of the cone lies on the axis of symmetry midway between its two ends A, B with the end A at a distance $a\cos\alpha$ from O.

    A thin uniform circular disc of the same surface density σ and radius $a\sin\beta$ is now fastened to the larger circular edge of the cone so that the centre of the disc is at B. Show that the composite body can rest in equilibrium with any point of the spherical surface on a horizontal floor provided that

    $\sin\alpha = \sqrt{1-\cos\beta}$.

16. Show by intergration that the centre of the gravity of the frustum obtained by cutting a uniform hollow hemisphereical shell of radius a and surface density σ by a plane parallel to its circular rim and at a distance $a\cos\alpha$ from the centre O is at the mid-point of OC where C is the centre of the smaller circular rim.

    A bowl is made by rigidly fixing the edge of a thin uniform circular plate of radius $a\sin a$ having the same surface density s to the smaller circular rim of the above frustum. Show that the centre of gravity the bowl is on OC at a distance $\left[\dfrac{1+\cos\alpha - \cos^2\alpha}{1+2\cos\alpha - \cos^2\alpha}\right] a\cos\alpha$ from O.

    Let $\alpha = \dfrac{\pi}{3}$ and let w be the weight of the bowl. A saucepan is made by rigidly fixing a thin uniform rod AB of length b and weight $\dfrac{w}{4}$ to the rim of the board as a handle such that the points O, A and B are collinear as shown in the figure. Find the position of the centre of gravity of the saucepan..

    The saucepan is freely suspended from the end B of the handle and hangs in equilibrium with the handle making an angle $\tan^{-1}\left(\dfrac{1}{7}\right)$ with the downward vertical. Show that $3b=4a$.